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Adaptive Control of MIMO Systems Using Sparsely Parameterized Controllers

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Abstract— **In applications, it is usually desirable to implement the simplest controller in terms of signal interconnections and controller order. In accordance with this goal, this paper presents an adaptive output-feedback control technique for MIMO systems, where the order and structure of each SISO entry of the controller can be assigned an arbitrary parameterization. The first parameterization is based on a MIMO input-output model of fixed window length. The second parameterization constrains all SISO entries of each row of the controller to have the same denominator of specified order; the third parameterization constrains all SISO entries of the controller to have the same denominator of specified order; and the fourth parameterization allows complete flexibility in setting the structure and order of each SISO entry of the controller. PID, infinite impulse-response, finite impulse-response, and sparsely parameterized controllers can be specified as special cases. This paper 1) defines the regressor structure for each parameterization; 2) enumerates the number of updated coefficients of each parameterization; 3) provides bounds on the McMillan degree of each parameterization; and 4) provides numerical examples to illustrate the relative effectiveness of each parameterization.**

I. Introduction

A well-known feature of dynamic output-feedback controllers based on the classical separation principle is the fact that the order of the controller is equal to the order of the plant. The same property applies to optimal controllers based on H_2 and H_∞ performance criteria. Consequently, the order of the controller is generically equal to the order of the plant. If, in addition, the command or disturbance is generated by Lyapunov-stable or unstable exogenous dynamics (representing, for example, steps, ramps, or sinusoids), then the internal model principle can be used to augment the plant with the exogenous dynamics, and thus the order of the resulting controller is greater than the plant order.

For plants of very high order, which occur, for example, in structural vibration, it is desirable to implement controllers of reduced order. Consequently, the literature on controller reduction is extensive [1]. An alternative approach to reducing a full-order controller is to directly construct a controller of fixed order [2], [3]. This nonconvex optimization problem is computationally demanding, however, and global convergence to the global minimizer is difficult to ensure.

Beyond controller reduction and fixed-order controller synthesis, it is often desirable in practice to construct controllers of fixed structure. For example, if a PID controller is required, then the PID gains can be determined within a fixed PID controller structure. Likewise, in MIMO applications, it may be desirable to seek a controller with internal structure in the sense that some entries—such as those that represent coupling paths of less importance—are of lower order than others. In some cases, it may desirable to simplify the controller by omitting certain controller entries, which can be done by constraining them to be zero, and thus obtaining sparse controllers [4]. In the extreme case, communication constraints may require a decentralized controller structure, where only the diagonal entries of a square MIMO controller are nonzero [5], [6]. Decentralized control is highly relevant to applications but remains a longstanding challenge in feedback control. Fixed-structure control problems such as decentralized control are typically challenging due to the nonconvexity of the underlying optimization problem. At the same time, because of computational constraints and implementation simplicity, control practitioners invariably seek the simplest possible controllers that meet performance and robustness criteria. These controllers are thus *fixed-gain, fixed-structure control laws.*

Although the literature on adaptive control is extensive [7]–[9], output-feedback adaptive control with arbitrary but fixed controller structure has not been considered. The contribution of the present paper is thus the development of an adaptive control technique for *MIMO controllers of fixed-but-arbitrary order and structure.* This technique for *adaptive-gain, fixed-structure control* is based on retrospective cost adaptive control (RCAC) [10]. RCAC is based on a retrospective cost function, which is optimized at each step to update the controller coefficients for use at the next step.

The contribution of the present paper is an extension of the controller structure used in [10] to provide greater flexibility in the implementation of RCAC. In particular, the controller structure used in [10] is a MIMO input-output model. The number of matrix coefficients in the input-output model is chosen by the user, and all entries are subject to optimization. We thus refer to the controller structure used in [10] as the *dense parameterization*. In the present paper, we consider three variations of the dense parameterization. In the second formulation, called the *diagonal denominator parameterization*, we constrain the structure of the MIMO input-output model so that its "denominator" is diagonal. This means, in effect, that each row of the MIMO transfer function has the same denominator. In the third formulation, called the *scalar denominator parameterization*, we constrain the structure of the MIMO input-output model so that its "denominator" is a polynomial multiplying the identity matrix. This parameterization is thus a specialization of the diagonal denominator parameterization. Finally, in the fourth formulation, called

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the *entrywise parameterization*, we constrain the structure of the MIMO input-output model so that each entry can have a different parameterization and order in terms of both its numerator and denominator. This means that a given entry could be chosen, for example, to be PID, infinite impulse response (IIR), or finite impulse response (FIR) of a specified order. Within the entrywise parameterization, it is possible to constrain chosen entries to be zero, and thus, sparsify the structure of the MIMO controller. Although the entrywise parameterization can be used to synthesize square MIMO controllers all of whose off-diagonal entries are zero, these controllers are not technically decentralized due to the use of a centralized information structure. A decentralized extension of RCAC is considered in [11] and RCAC was applied using entrywise parameterization in [12].

II. Problem Formulation

Consider a discrete-time system modeled by

$$x_{k+1} = Ax_k + Bu_k + Bw_k, \quad y_{0,k} = Cx_k + v_k, \tag{1}$$

$$y_k = y_{0,k} + v_k, \quad z_k \triangleq r_k - y_k, \tag{2}$$

where $k \geq 0$ is the step, $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^{l_u}$ is the control input, $w_k \in \mathbb{R}^{l_u}$ is the disturbance, $y_{0,k} \in \mathbb{R}^{l_y}$ is the plant output, $y_k \in \mathbb{R}^{l_y}$ is the measurement, $r_k \in \mathbb{R}^{l_y}$ is the command, $v_k \in \mathbb{R}^{l_y}$ is the sensor noise, and the command-following error $z_k \in \mathbb{R}^{l_z}$ is the performance variable. Note that $l_z = l_y$.

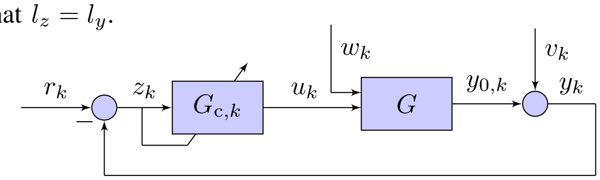

Fig. 1: Block diagram of the adaptive servo problem with the adaptive controller $G_{c,k}$ and plant G.

The goal is to develop an adaptive output-feedback controller

$$u_k = G_{c,k}(\mathbf{q})z_k, \tag{3}$$

where $\mathbf{q}$ is the forward-shift operator, and $G_{c,k}$ is a time-varying matrix of strictly proper, rational polynomials in $\mathbf{q}$ that minimizes z_k in the presence of the disturbance signal d_k with limited modeling information about (1), (2). Note that each entry of $G_{c,k}$ can be represented as a time-domain transfer function at each step k.

III. Controller Parameterizations

We write (3) as

$$u_k = \phi_k\theta_k, \tag{4}$$

where $\phi_k \in \mathbb{R}^{l_u \times l_\theta}$ is the regressor matrix constructed using the past error measurements z_k and control inputs u_k, and $\theta_k \in \mathbb{R}^{l_\theta}$ is the vector of controller coefficients. There are multiple ways to parameterize (3) as (4). In this paper, we present and compare four such parameterizations of $G_{c,k}$. Namely, dense parameterization (DP), diagonal denominator parameterization (DDP), scalar denominator parameterization (SDP), and entrywise parameterization (EP). Table I lists salient features of each parameterization. As all four controller parameterizations are expressed as (4), RCAC presented in [10] can be used to adapt the controller coefficients.

A. Dense Parameterization (DP)

Consider a controller constructed as a strictly proper input-output model with an l_c-step data window, such that the control u_k is given by

$$u_k = \sum_{i=1}^{l_c} P_{i,k}u_{k-i} + \sum_{i=1}^{l_c} Q_{i,k}z_{k-i}, \tag{5}$$

where $P_{i,k} \in \mathbb{R}^{l_u \times l_u}$ and $Q_{i,k} \in \mathbb{R}^{l_u \times l_z}$ are the fully-populated controller coefficient matrices. The parameterization in (5) is called dense since the denominator coefficient matrices $P_{i,k}$ are nonsparse. In terms of the forward-shift operator $\mathbf{q}$, the $l_u \times l_z$ controller transfer function $G_{c,k}$ from z_k to u_k is given by

$$G_{c,k}(\mathbf{q}) = \left(I_{l_u}\mathbf{q}^{l_c} - P_{1,k}\mathbf{q}^{l_c-1} - \cdots - P_{l_c,k}\right)^{-1} \cdot \left(Q_{1,k}\mathbf{q}^{l_c-1} + \cdots + Q_{l_c,k}\right). \tag{6}$$

Equation (5) can be expressed as (4) by defining

$$\theta_k \triangleq \text{vec}\,[P_{1,k} \ \cdots \ P_{l_c,k} \ Q_{1,k} \ \cdots \ Q_{l_c,k}] \in \mathbb{R}^{l_\theta}, \tag{7}$$

$$\phi_k \triangleq \begin{bmatrix} u_{k-1} \\ \vdots \\ u_{k-l_c} \\ z_{k-1} \\ \vdots \\ z_{k-l_c} \end{bmatrix}^{\mathrm{T}} \otimes I_{l_u} \in \mathbb{R}^{l_u \times l_\theta}, \tag{8}$$

where $l_\theta \triangleq l_c l_u(l_u + l_z)$.

Let $\bar{p}_k(\mathbf{q})$ denote the least common denominator of $G_{c,k}(\mathbf{q})$ in (6). It follows from [13, p. 522] that the order of $\bar{p}_k(\mathbf{q})$ is bounded from above by $l_c l_u$. Consequently, the McMillan degree of (5) is bounded from above by $l_c l_u l_z$.

B. Diagonal Denominator Parameterization (DDP)

Next, consider a controller constructed as a strictly proper input-output model with l_c-step data window, such that the control u_k is given by

$$u_k = \sum_{i=1}^{l_c} P_{i,k}u_{k-i} + \sum_{i=1}^{l_c} Q_{i,k}z_{k-i}, \tag{9}$$

where $P_{i,k} \in \mathbb{R}^{l_u \times l_u}$ and $Q_{i,k} \in \mathbb{R}^{l_u \times l_z}$ are the controller coefficient matrices, and $P_{i,k}$ are diagonal.

In terms of the forward-shift operator $\mathbf{q}$, the controller transfer function G_c from z_k to u_k is given by

$$G_{c,k}(\mathbf{q}) = \left(I_{l_u}\mathbf{q}^{l_c} - P_{1,k}\mathbf{q}^{l_c-1} - \cdots - P_{l_c,k}\right)^{-1} \cdot \left(Q_{1,k}\mathbf{q}^{l_c-1} + \cdots + Q_{l_c,k}\right). \tag{10}$$

Note that the denominator matrix in (10) is diagonal, and thus all entries in each row of G_c have the same poles.

Parameterization	Controller coefficient size l_θ	Maximum order of each SISO entry	Maximum McMillan degree	Remarks
Dense	$l_{\rm c}l_u^2+l_{\rm c}l_ul_z$	$l_{\rm c}l_u$	$l_{\rm c}l_ul_z$	Same denominator in all entries
Diagonal Denominator	$l_{\rm c}l_u+l_{\rm c}l_ul_z$	$l_{\rm c}$	$l_{\rm c}l_ul_z$	Same denominator in all entries of each row
Scalar Denominator	$l_{\rm c}+l_{\rm c}l_ul_z$	$l_{\rm c}$	$l_{\rm c}l_z$	Same denominator in all entries
Entrywise	$\sum_{i=1}^{l_u}\sum_{j=1}^{l_z}2l_{ij}$	l_{ij}	$l_z\sum_{i=1}^{l_u}\sum_{j=1}^{l_z}l_{ij}$	Different denominator in each entry

TABLE I: Properties of the four controller parameterizations. The bound on the MIMO McMillan degree for each parameterization is based on the order of a possibly nonminimal state space realization of $G_{\rm c}$.

We write (9) as

$$u_k=\phi_k\theta_k=\phi_{u,k}\theta_{u,k}+\phi_{z,k}\theta_{z,k}, \quad (11)$$

where

$$\phi_k\triangleq\begin{bmatrix}\phi_{u,k} & \phi_{z,k}\end{bmatrix}\in\mathbb{R}^{l_u\times\left(l_ul_{\rm c}+l_ul_zl_{\rm c}\right)}, \quad (12)$$

$$\phi_{u,k}\triangleq\begin{bmatrix}{\rm dg}u_{k-1} & \cdots & {\rm dg}u_{k-l_{\rm c}}\end{bmatrix}\in\mathbb{R}^{l_u\times l_ul_{\rm c}}, \quad (13)$$

$$\phi_{z,k}\triangleq\begin{bmatrix}z_{k-1}\\ \vdots\\ z_{k-l_{\rm c}}\end{bmatrix}^{\rm T}\otimes I_{l_u}\in\mathbb{R}^{l_u\times l_ul_zl_{\rm c}},\ \theta_k\triangleq\begin{bmatrix}\theta_{u,k}\\ \theta_{z,k}\end{bmatrix}\in\mathbb{R}^{l_\theta}, \quad (14)$$

$$\theta_{u,k}\triangleq{\rm vec}\begin{bmatrix}{\rm dg}^{-1}P_{1,k} & \cdots & {\rm dg}^{-1}P_{l_{\rm c},k}\end{bmatrix}\in\mathbb{R}^{l_ul_{\rm c}}, \quad (15)$$

$$\theta_{z,k}\triangleq{\rm vec}\begin{bmatrix}Q_{1,k} & \cdots & Q_{l_{\rm c},k}\end{bmatrix}\in\mathbb{R}^{l_ul_zl_{\rm c}}, \quad (16)$$

${\rm dg}\colon\mathbb{R}^{l_x}\to\mathbb{R}^{l_x\times l_x}$ maps a vector into a diagonal-square matrix with diagonal entries of the vector, ${\rm dg}^{-1}\colon\mathbb{R}^{l_x\times l_x}\to\mathbb{R}^{l_x}$ maps a square matrix into a vector of its diagonal entries, and $l_\theta\triangleq l_ul_{\rm c}+l_ul_zl_{\rm c}$.

Let $\bar{p}_k(\mathbf{q})$ denote the least common denominator of $G_{{\rm c},k}(\mathbf{q})$ in (10). It follows from [13, p. 522] that the order of $\bar{p}_k(\mathbf{q})$ is bounded from above by $l_{\rm c}l_u$. Consequently, the McMillan degree of (9) is bounded from above by $l_{\rm c}l_ul_z$.

C. Scalar Denominator Parameterization (SDP)

Next, consider a controller constructed as a strictly proper input-output model with $l_{\rm c}$-step data window, such that the control u_k is given by

$$u_k=\sum_{i=1}^{l_{\rm c}}p_{i,k}u_{k-i}+\sum_{i=1}^{l_{\rm c}}Q_{i,k}z_{k-i}, \quad (17)$$

where $p_{i,k}\in\mathbb{R}$, and $Q_{i,k}\in\mathbb{R}^{l_u\times l_z}$ are the controller coefficients.

In terms of the forward-shift operator $\mathbf{q}$, the controller transfer function $G_{\rm c}$ from z to u is given by

$$G_{{\rm c},k}(\mathbf{q})=\frac{Q_{1,k}\mathbf{q}^{l_{\rm c}-1}+\cdots+Q_{l_{\rm c},k}}{\mathbf{q}^{l_{\rm c}}-p_{1,k}\mathbf{q}^{l_{\rm c}-1}-\cdots-p_{l_{\rm c},k}}. \quad (18)$$

We write (17) as

$$u_k=\phi_k\theta_k=\phi_{u,k}\theta_{u,k}+\phi_{z,k}\theta_{z,k}, \quad (19)$$

where

$$\phi_k\triangleq\begin{bmatrix}\phi_{u,k} & \phi_{z,k}\end{bmatrix}\in\mathbb{R}^{l_u\times\left(l_{\rm c}+l_ul_zl_{\rm c}\right)}, \quad (20)$$

$$\phi_{u,k}\triangleq\begin{bmatrix}u_{k-1} & \cdots & u_{k-l_{\rm c}}\end{bmatrix}\in\mathbb{R}^{l_u\times l_{\rm c}}, \quad (21)$$

$$\phi_{z,k}\triangleq\begin{bmatrix}z_{k-1}\\ \vdots\\ z_{k-l_{\rm c}}\end{bmatrix}^{\rm T}\otimes I_{l_u}\in\mathbb{R}^{l_u\times l_ul_zl_{\rm c}},\ \theta_k\triangleq\begin{bmatrix}\theta_{u,k}\\ \theta_{z,k}\end{bmatrix}\in\mathbb{R}^{l_\theta}, \quad (22)$$

$$\theta_{u,k}\triangleq\begin{bmatrix}p_{1,k} & \cdots & p_{l_{\rm c},k}\end{bmatrix}^{\rm T}\in\mathbb{R}^{l_{\rm c}}, \quad (23)$$

$$\theta_{z,k}\triangleq{\rm vec}\begin{bmatrix}Q_{1,k} & \cdots & Q_{l_{\rm c},k}\end{bmatrix}\in\mathbb{R}^{l_ul_zl_{\rm c}}, \quad (24)$$

and $l_\theta\triangleq l_{\rm c}+l_ul_zl_{\rm c}$.

Let $\bar{p}_k(\mathbf{q})$ denote the least common denominator of $G_{{\rm c},k}(\mathbf{q})$ in (18). It follows from [13, p. 522] that the order of $\bar{p}_k(\mathbf{q})$ is bounded from above by $l_{\rm c}$. Consequently, the McMillan degree of (17) is bounded from above by $l_{\rm c}l_z$.

D. Entrywise Parameterization (EP)

Finally, we consider an adaptive controller such that

$$G_{{\rm c},k}(\mathbf{q})\triangleq\begin{bmatrix}G_{{\rm c},11,k}(\mathbf{q}) & \ldots & G_{{\rm c},1l_z,k}(\mathbf{q})\\ \vdots & \ddots & \vdots\\ G_{{\rm c},l_u1,k}(\mathbf{q}) & \ldots & G_{{\rm c},l_ul_z,k}(\mathbf{q})\end{bmatrix}, \quad (25)$$

where each entry $G_{{\rm c},ij,k}$ of the $l_u\times l_z$ transfer function $G_{{\rm c},k}$ is an IIR or FIR controller of order l_{ij}, or a PID controller. For each SISO controller entry $G_{{\rm c},ij}$ we define

$$u_{ij,k}\triangleq G_{{\rm c},ij,k}(\mathbf{q})z_{j,k}, \quad (26)$$

and construct $\phi_{ij,k}\in\mathbb{R}^{l_{\theta,ij}}$,

$$\phi_{ij,k}\triangleq\begin{bmatrix}z_{j,k-1}\\ \sum_{p=1}^{k-1}z_{j,p}\\ z_{j,k-1}-z_{j,k-2}\end{bmatrix},\phi_{ij,k}\triangleq\begin{bmatrix}u_{ij,k-1}\\ \vdots\\ u_{ij,k-l_{ij}}\\ z_{j,k-1}\\ \vdots\\ z_{j,k-l_{ij}}\end{bmatrix}, \quad (27)$$

$$\phi_{ij,k}\triangleq\begin{bmatrix}z_{j,k-1}\\ \vdots\\ z_{j,k-l_{ij}}\end{bmatrix}, \quad (28)$$

for a PID, IIR, or FIR $G_{{\rm c},ij,k}(\mathbf{q})$, respectively, where

$$l_{\theta,ij}=\begin{cases}3, & \text{PID } G_{{\rm c},ij,k}(\mathbf{q}),\\ 2l_{ij}, & \text{IIR } G_{{\rm c},ij,k}(\mathbf{q}),\\ l_{ij}, & \text{FIR } G_{{\rm c},ij,k}(\mathbf{q}).\end{cases} \quad (29)$$

Additionally, we define

$$\phi_{i,k} \triangleq \begin{bmatrix} \phi_{i1,k} \\ \vdots \\ \phi_{il_z,k} \end{bmatrix} \in \mathbb{R}^{l_{\theta_i}}, \quad \theta_{i,k} \triangleq \begin{bmatrix} \theta_{i1,k} \\ \vdots \\ \theta_{il_z,k} \end{bmatrix} \in \mathbb{R}^{l_{\theta_i}}, \quad (30)$$

where $l_{\theta_i} \triangleq \sum_{j=1}^{l_z} l_{\theta,ij}$. Finally, define

$$\phi_k \triangleq \begin{bmatrix} \phi_{1,k}^{\mathrm{T}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \phi_{l_u,k}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{l_u \times l_\theta}, \theta_k \triangleq \begin{bmatrix} \theta_{1,k} \\ \vdots \\ \theta_{l_z,k} \end{bmatrix} \in \mathbb{R}^{l_\theta}, \quad (31)$$

where $l_\theta \triangleq \sum_{i=1}^{l_u} \sum_{j=1}^{l_z} l_{\theta,ij}$. Note that fixed-structure controller parameterization allows us to separately specify the structure of each entry (25) of G_{c} through the construction of ϕ_k.

Let $\bar{p}_k(\mathbf{q})$ denote the least common denominator of $G_{\mathrm{c},k}(\mathbf{q})$ in (25). It follows from [13, p. 522] that the order of $\bar{p}_k(\mathbf{q})$ is bounded from above by $\sum_{i=1}^{l_u} \sum_{j=1}^{l_z} l_{ij}$. Consequently, the McMillan degree of (25) is bounded from above by $\sum_{i=1}^{l_u} \sum_{j=1}^{l_z} l_{ij} l_z$.

Note that, because $D(\mathbf{q})$ in EP is not diagonal, the coefficients of the MIMO transfer function are not directly optimized by RCAC. In contrast, in DDP, SDP, and EP, RCAC directly updates each numerator and denominator coefficient of G_{c}. Table I summarizes the key features of the parameterizations discussed above.

To illustrate the parameterizations, consider the MIMO transfer function

$$G_{\mathrm{c}}(\mathbf{q}) = \begin{bmatrix} \frac{1}{\mathbf{q}+1} & \frac{1}{\mathbf{q}+2} \\ \frac{1}{\mathbf{q}+3} & \frac{1}{\mathbf{q}+4} \end{bmatrix}, \quad (32)$$

which is written in entrywise parameterization, where $l_{\mathrm{c}} = 1$ and $l_\theta = 2 l_u l_z l_{\mathrm{c}} = 8$. The diagonal denominator parameterization of (32) is

$$G_{\mathrm{c}}(\mathbf{q}) = \begin{bmatrix} \mathbf{q}^2+3\mathbf{q}+2 & 0 \\ 0 & \mathbf{q}^2+7\mathbf{q}+12 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}+2 & \mathbf{q}+1 \\ \mathbf{q}+4 & \mathbf{q}+3 \end{bmatrix}, \quad (33)$$

where $l_{\mathrm{c}} = 2$ and $l_\theta = l_{\mathrm{c}} l_u + l_u l_z l_{\mathrm{c}} = 12$. The scalar denominator parameterization of (32) is

$$G_{\mathrm{c}}(\mathbf{q}) = \frac{1}{\mathbf{q}^4+10\mathbf{q}^3+35\mathbf{q}^2+50\mathbf{q}+24} \cdot \begin{bmatrix} \mathbf{q}^3+9\mathbf{q}^2+26\mathbf{q}+24 & \mathbf{q}^3+8\mathbf{q}^2+19\mathbf{q}+12 \\ \mathbf{q}^3+7\mathbf{q}^2+14\mathbf{q}+8 & \mathbf{q}^3+6\mathbf{q}^2+11\mathbf{q}+6 \end{bmatrix}, \quad (34)$$

where $l_{\mathrm{c}} = 4$ and $l_\theta = l_{\mathrm{c}} + l_u l_z l_{\mathrm{c}} = 20$. Finally, the dense parameterization of (32) is

$$G_{\mathrm{c}}(\mathbf{q}) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad (35)$$

where

$$N_{11}(\mathbf{q}) \triangleq \frac{2}{3}\mathbf{q}^3 + 5\mathbf{q}^2 - \frac{34}{3}\mathbf{q} + 8, \quad (36)$$

$$N_{12}(\mathbf{q}) \triangleq \frac{2}{3}\mathbf{q}^3 + 5\mathbf{q}^2 - \frac{34}{3}\mathbf{q} + 7, \quad (37)$$

$$N_{21}(\mathbf{q}) \triangleq -2\mathbf{q}^4 - 20\mathbf{q}^3 - 70\mathbf{q}^2 - 100\mathbf{q} - 48, \quad (38)$$

$$N_{22}(\mathbf{q}) \triangleq -2\mathbf{q}^4 - 20\mathbf{q}^3 - 70\mathbf{q}^2 - 100\mathbf{q} - 48, \quad (39)$$

$$D_{11}(\mathbf{q}) \triangleq -\frac{1}{3}\mathbf{q}^5 - \frac{7}{2}\mathbf{q}^4 - \frac{40}{3}\mathbf{q}^3 - \frac{45}{2}\mathbf{q}^2 - \frac{49}{3}\mathbf{q} - 4, \quad (40)$$

$$D_{12}(\mathbf{q}) \triangleq \frac{1}{3}\mathbf{q}^5 + \frac{29}{6}\mathbf{q}^4 + \frac{80}{3}\mathbf{q}^3 + \frac{415}{6}\mathbf{q}^2 + \frac{83}{3}\mathbf{q} + 36, \quad (41)$$

$$D_{21}(\mathbf{q}) \triangleq \mathbf{q}^6 + 13\mathbf{q}^5 + 67\mathbf{q}^4 + 175\mathbf{q}^3 + 244\mathbf{q}^2 + 172\mathbf{q} + 48, \quad (42)$$

$$D_{22}(\mathbf{q}) \triangleq -\mathbf{q}^6 - 17\mathbf{q}^5 - 117\mathbf{q}^4 - 415\mathbf{q}^3 - 794\mathbf{q}^2 - 768\mathbf{q} - 288, \quad (43)$$

where $l_{\mathrm{c}} = 6$ and $l_\theta = l_{\mathrm{c}} l_u^2 + l_u l_z l_{\mathrm{c}} = 48$.

This example shows that, for a given controller transfer function G_{c}, the dense, diagonal denominator, scalar denominator, and entrywise parameterizations of G_{c} may be different in terms of the data window l_{c} and the number of coefficients l_θ that are optimized. In particular, the McMillan degree of G_{c} given by (32) is 4, which means that a minimal state space realization of G_{c} has 4 states. For EP, DDP, SDP, and DP, l_{c} is given by 1, 2, 4, and 6, respectively, and l_θ is given by 8, 12, 20, and 48, respectively. These numbers show that EP is the most efficient representation of G_{c} given by (32). However, this observation is specific to G_{c} given by (32), since DP with $l_{\mathrm{c}} = 6$ and $l_\theta = 48$ can represent a much larger class of 2×2 transfer functions than EP can represent with $l_{\mathrm{c}} = 1$ and $l_\theta = 8$.

Each of the four parameterizations of an $l_u \times l_z$ controller transfer function is represented as an input-output model with an l_{c}-step data window. This is in contrast to a state space controller representation $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$, where A_{c} is of size $n_{\mathrm{c}} \times n_{\mathrm{c}}$. If $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ is minimal, then n_{c} is the McMillan degree of the transfer function corresponding to $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ [14, p. 319].

Although the relationship between the order l_{c} and the McMillan degree of a controller in state space form is straightforward, the relationship is more complicated for a controller in input-output model form. Writing (32) as

$$G_{\mathrm{c}}(\mathbf{q}) = D_{\mathrm{c}}(\mathbf{q})^{-1} N_{\mathrm{c}}(\mathbf{q}), \quad (44)$$

it follows that, if D_{c} and N_{c} are left coprime, then the McMillan degree of G_{c} is given by $\deg\det D_{\mathrm{c}}$. Note, however, that, for the scalar denominator parameterization (34), $\deg\det D_{\mathrm{c}} = 16$, which is not equal to the McMillan degree of G_{c}, and thus (34) is not left coprime.

IV. Numerical Examples

In this section we compare the controller complexity needed to achieve step-command following for a 2×2 system using LQG and RCAC with each of the four parameterizations described in the preceding section. In particular, we consider two oscillators connected in series with springs and dampers, as shown in Figure 2. Forces f_a, f_b can be applied on masses m_a, m_b, respectively, and positions measurements

q_a, q_b of the two masses m_a, m_b are available. Thus, the system has two inputs and outputs. The objective is to follow setpoint commands to q_a, q_b.

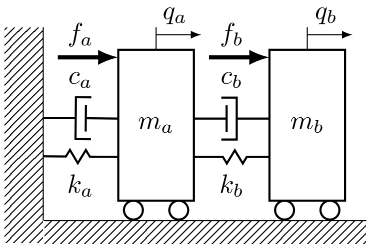

Fig. 2: Series dual-mass oscillator. Position measurements q_a, q_b of the masses m_a, m_b are available. Forces f_a and f_b are applied to masses m_a and m_b, respectively.

	m (kg)	c (N/m^2)	k (N/m)
a	1	7	4
b	0.8	3.5	10

TABLE II: Values of coefficients used to construct the oscillator model.

For the values of masses, stiffnesses, and damping coefficients listed in Table II an exact discretization of the continuous-time system using zero-order hold with a sample rate of 5 Hz is given by (1), (2) where

$$A \triangleq \begin{bmatrix} 0.8777 & 0.0847 & 0.0777 & 0.0338 \\ -0.7630 & 0.1363 & 0.4241 & 0.2262 \\ 0.1436 & 0.0423 & 0.8429 & 0.1333 \\ 1.0741 & 0.2827 & -1.2433 & 0.4077 \end{bmatrix}, \tag{45}$$

$$B \triangleq \begin{bmatrix} 0.0112 & 0.0034 \\ 0.0847 & 0.0423 \\ 0.0034 & 0.0191 \\ 0.0423 & 0.1666 \end{bmatrix}, \; C \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{46}$$

$$x_k \triangleq \begin{bmatrix} q_{a,k} & \dot{q}_{a,k} & q_{b,k} & \dot{q}_{b,k} \end{bmatrix}^{\mathrm{T}}, \tag{47}$$

$$u_k \triangleq \begin{bmatrix} f_{a,k} & f_{b,k} \end{bmatrix}^{\mathrm{T}}. \tag{48}$$

We specify $w_k \sim (0, 0.01^2), v_k \sim (0, 0.001^2)$, $x_0 = 0$, and the command is

$$r_k = \begin{cases} \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}, & t < 50 \text{ sec}, \\ \begin{bmatrix} -1 & -3 \end{bmatrix}^{\mathrm{T}}, & t \geq 50 \text{ sec}. \end{cases} \tag{49}$$

Example 1. *Discrete-time LQG controller.* A discrete-time LQG controller with integral action is designed for the plant using the MATLAB command lqg with the weights

$$Q_{xu} = I_6, \quad Q_{wv} = I_6, \quad Q_{\mathrm{i}} = 100 I_2. \tag{50}$$

The resulting controller is given by

$$x_{\mathrm{c},k+1} = A_{\mathrm{c}} x_{\mathrm{c},k} + B_{\mathrm{c}} z_k, \tag{51}$$

$$u_k = C_{\mathrm{c}} x_{\mathrm{c},k}, \tag{52}$$

where $x_{\mathrm{c},k} \in \mathbb{R}^6$, $A_{\mathrm{c}} \in \mathbb{R}^{6\times6}$, $B_{\mathrm{c}} \in \mathbb{R}^{6\times2}$, and $C_{\mathrm{c}} \in \mathbb{R}^{2\times6}$. As shown in Figure 3, step-command following is achieved. $\diamond$

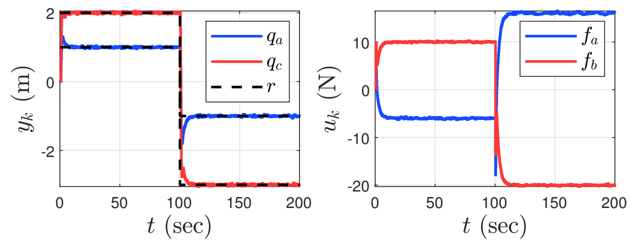

Fig. 3: Example 1: Step-command following using a discrete-time LQG controller with integral action.

Example 2. *RCAC with dense parameterization.* RCAC is applied with dense parameterization with $l_{\mathrm{c}} = 3, P_0 = 10^3 I, R_u = 10^{-5} I$. The number of controller coefficients adapted is 24. The maximum achievable McMillan degree of the adaptive controller is $l_u l_z l_{\mathrm{c}} = 12$. For this and all examples that follow we set $G_{\mathrm{f}}(\mathbf{q}) = -\frac{1}{\mathbf{q}} CB$. As shown in Figure 4, step-command following is achieved. $\diamond$

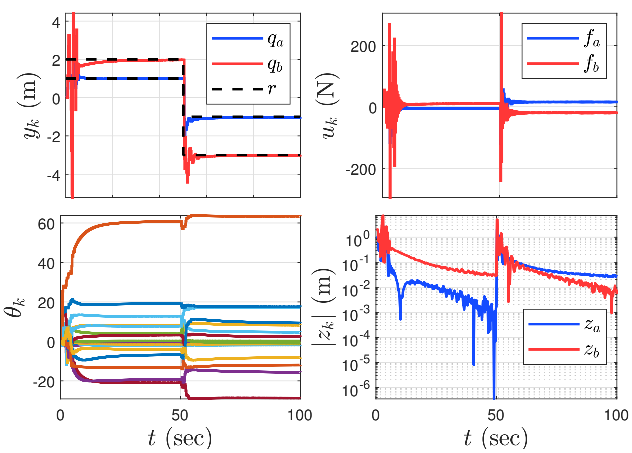

Fig. 4: Example 2: Step-command following with RCAC using the dense parameterization.

Example 3. *RCAC with diagonal denominator parameterization.* RCAC is applied with diagonal denominator parameterization with $l_{\mathrm{c}} = 7, P_0 = 10^3 I, R_u = 10^{-5} I$. The number of controller coefficients adapted is 18. The maximum achievable McMillan degree of the adaptive controller is $l_u l_z l_{\mathrm{c}} = 28$. As shown in Figure 5, step-command following is achieved. $\diamond$

Example 4. *RCAC with scalar denominator parameterization.* RCAC is applied with scalar denominator parameterization with $l_{\mathrm{c}} = 8, P_0 = 10^3 I, R_u = 10^{-5} I$. The number of controller coefficients adapted is 35. The maximum achievable McMillan degree of the adaptive controller is $l_z l_{\mathrm{c}} = 16$. As shown in Figure 6, step-command following is achieved. $\diamond$

Example 5. *RCAC with entrywise parameterization.* RCAC is applied with entrywise parameterization with $P_0 = 0.1 I, R_u = 0$, where G_{c} has the structure

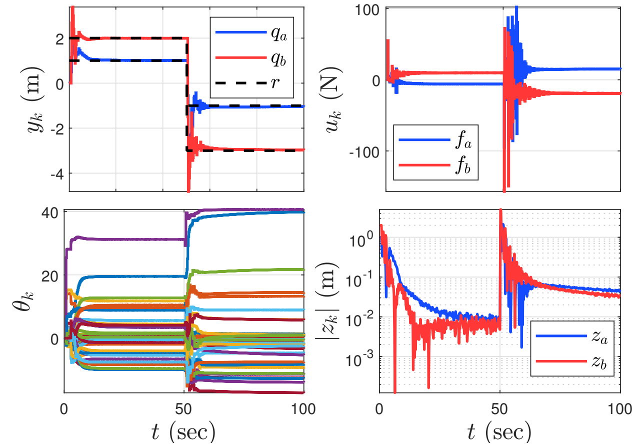

Fig. 5: Example 3: Step-command following with RCAC using the diagonal denominator parameterization.

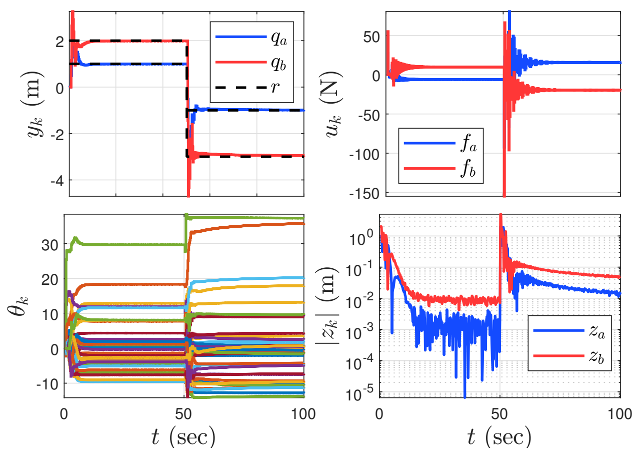

Fig. 6: Example 4: Step-command following with RCAC using the scalar denominator parameterization.

$$G_{\mathrm{c},k}(\mathbf{q}) = \begin{bmatrix} \text{PID} & \text{6th-order IIR} \\ 0 & \text{PID} \end{bmatrix}. \tag{53}$$

The number of controller coefficients adapted is 18. The maximum achievable McMillan degree of the adaptive controller is $l_z \sum_{i=1}^{l_u} \sum_{j=1}^{l_z} l_{ij} = 24$. As shown in Figure 7, step-command following is achieved. $\diamond$

V. Discussion and Conclusions

This paper presented four parameterizations for MIMO controllers—dense (DP), diagonal denominator (DDP), scalar denominator (SDP), and entrywise (EP)—that can be used with retrospective cost adaptive control (RCAC). The three new parameterizations, which constitute the contribution of this paper, provide greater flexibility in constraining the internal structure of the compensator. For a fixed number of optimized parameters, each parameterization can potentially produce a controller of different McMillan degree. To compare the efficacy of these parameterizations, all four parameterizations were applied to a setpoint command-following problem for a two-input, two-output, LTI system. Similar performance was obtained for DP, DDP, SDP, and EP using 24, 18, 35, and 18 parameters, respectively.

For a sufficiently large data-window size l_c, each parameterization can represent a controller of arbitrary McMillan

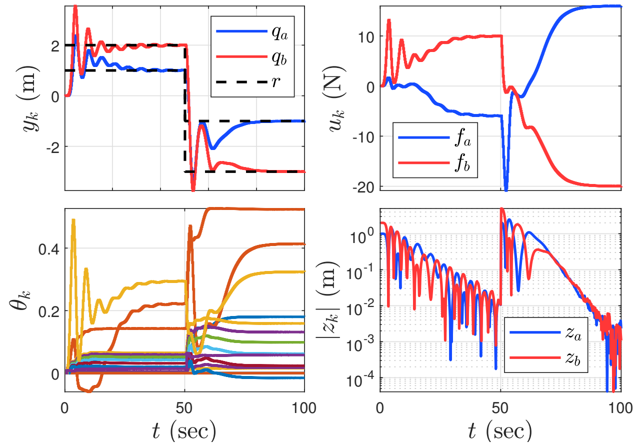

Fig. 7: Example 5: Step-command following with RCAC using the entrywise parameterization.

degree. However, the required value of l_c depends on the choice of parameterization. In addition, since l_θ also depends on the choice of parameterization, it is of interest to compare the performance of DP, DDP, SDP, and EP for the same value of l_θ. A key question for future research is thus to determine which parameterization is the most effective for a given number of controller parameters. A related question concerns the class of controllers that are achievable by each parameterization for a given value of l_c. The examples in this paper suggest that EP is potentially the most effective since it allows independent poles in each entry of G_c, and thus the largest achievable McMillan degree for a given number of controller parameters.

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