Transmission Matrices for Physical-System Modeling

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Transfer functions model input–output relationships in linear systems. For example, viewing force as an input, velocity can be seen as an output. Likewise, voltage from a battery gives rise to current flow. The gain, poles, and zeros of these transfer functions provide insight into the response to inputs.

Transfer functions can be derived by applying the laws of dynamics and circuits, such as Newton's and Kirchhoff's laws. An alternative approach to obtaining them is to develop models of mechanical and electrical components and connect them by accounting for power (the flow of energy) by means of *power-conjugate variables* (variables whose product is power). In mechanical systems, force and velocity are power-conjugate variables; in electrical systems, voltage and current are power-conjugate variables.

Input–output models can be used to relate effort and flow variables, which are power conjugate. In structural modeling, power-conjugate variables are force and velocity in translational motion and moment and angular velocity in rotational motion; in electrical systems, they are voltage and current, respectively; in fluid and acoustic systems, they are pressure and volume velocity; and in thermodynamic systems, they are temperature difference and entropy flow rate.

Since power-conjugate variables occur in pairs, they are connected by *power transmission matrices* (*PTMs*), which are 2×2 transfer functions. PTMs have a variety of forms and names arising from the way the variables are defined and ordered. They are called *four-terminal structures*, *four-terminal networks*, *quadripole*, *transfer matrices*, and *two-port networks*. Depending on the choice of input and output signals, the entries of the matrices are known as ABCD parameters, cascade parameters, chain parameters, four-pole parameters, impedance parameters, admittance parameters, hybrid parameters, inverse hybrid parameters, scattering parameters, scattering transfer parameters, and transmission parameters [1, pp. 28–31], [2, pp. 69, 70, 315–317], [3, pp. 56–58], [4, pp. 243–257], [5, pp. 3–15], [6, pp. 312–330], [7, pp. 10-35–10-37], [8, pp. 132–134], [9, pp. 16–32] [10, pp. 49–57], [11, Ch. 3], [12, pp. 208–214], [13, Ch. 12], [14, Ch. 11], [15, Ch. 11], [16, pp. 24–27]. Reference [17] includes detailed definitions and further references for these matrices. Transformations among six versions of these parameters are given in [6, p. 317], where the **b** parameters are most closely related to power transmission matrices for electrical circuits as they are defined in the present article.

PTMs and their variants have been used in diverse mechanical and electrical applications. PTMs are employed in [11] to derive equations of motion for lumped mechanical systems as well as continuum structures, such as shafts and beams. For mechanical applications, the use of PTMs circumvents the necessity of free-body analysis and eliminates the need to determine reaction forces and torques. On the electrical side, PTMs are used to model distributed networks, including transmission lines [3], [5, Ch. 1, pp. 200–218], [12, pp. 212–214], [13, pp. 361–362].

As described in "Summary," a benefit of power transmission matrices is that analogous models can be derived

Summary

Power transmission matrices (PTMs) for structures are 2×2 matrices that relate force and velocity at one terminal to force and velocity at another terminal. This technique provides an elegant approach to deriving transfer functions for structures consisting of masses, springs, and dashpots, that is, dampers. PTMs were developed during the 1970s and 1980s, but they are rarely mentioned today except briefly in textbooks. The first goal of this tutorial is to highlight the utility of this modeling technique. A related aim is to bring inerters into this framework. The article revisits the classical topic of analogies by deriving PTMs for two-port networks. The across-through analogy is used to construct circuits whose transfer functions match those of the analogous structure. This article is intended for all students and practitioners of control systems who may benefit from awareness of this modeling technique.

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across domains of application. The study of analogies between mechanical and electrical systems has a long history. The classic book [18], which remains the unique reference devoted to the topic, encompasses mechanical, electrical, and acoustic systems. Circuit models for structures are discussed within the context of analog computation in [19, Ch. 9]. A recent survey [20] provides a detailed history of the development of electromechanical analogies.

In an electrical circuit, potential can be defined relative to a constant potential level, such as an *earth ground*. Analogously, the velocity of a point in a structure can be defined relative to a point that has constant inertial velocity, for example, a point on an inertially nonrotating massive body [21]. Such a point provides an *inertial ground.* Voltage and velocity are relative (across) variables, whereas force and current are absolute (through) variables [22, pp. 18–22], [23, p. 20], [24, pp. 45–47], [25, p. 21], [26, pp. 39–58]. In electrical applications, an earth ground provides an approximately infinite volume that can transfer electrons without losing its charge neutrality. Analogously, in structural applications, an inertially nonrotating massive body has approximately infinite mass and does not accelerate translationally or rotationally in response to reaction forces and torques. A physical terminal is a physical point of attachment, such as the wires emanating from a resistor or the ends of a spring, while a reference terminal is an inertial or earth ground [27], [28].

There are two competing analogies between electrical and mechanical systems, denoted in this article by *fE-vI* and *fI-vE*, each of which (for historical reasons) has at least four names. For example, *fE-vI* is called the Maxwell analogy, direct analogy, impedance analogy, and effort-flow analogy, and it associates force with potential and velocity with current. On the other hand, *fI-vE* is known as the Firestone analogy, inverse analogy, mobility analogy, and across-through analogy, and it pairs force with current and velocity with potential. Early papers [29]–[31] discuss various aspect of these analogies. With voltage and current taken as analogous to velocity and force, respectively, we note that mass m, damping c, and stiffness k are related to capacitance C, resistance R, and inductance L by $m = C$, $c = 1/R$, and $k = 1/L$. Observe that capacitors and masses store energy provided by the across variables voltage and velocity, respectively, whereas inductors and springs store energy provided by the through variables current and force, respectively.

Bond graph modeling is based on the effort-and-flow analogy *fE-vI* [2, pp. 343–418], [32, pp. 260–270], [33], [34], [35, p. 123–167], [25, pp. 113–129]. Bond graphs are related to transmission matrices that are based on *fE-vI* in [2, pp. 69, 70, 315–317]. In contrast, the present article focuses on *fI-vE*; the reason for this choice is the analogy between inertial and earth grounds as well as the preservation of series and parallel connections across domains.

This article encompasses the inerter as a distinct mechanical component [36]–[38], in which the reaction force is proportional to the relative acceleration of the physical terminals rather than the inertial acceleration of a mass as explained in "What Is an Inerter?" Mechanical networks can be built from four components (masses, inerters, springs, and dashpots), whereas electrical networks can be constructed from only three (resistors, inductors, and capacitors). Some brief remarks are provided on the

What Is an Inerter?

The relationship between the acceleration of a mass and the force applied to the mass is given by Newton's second law. The acceleration vector is the second derivative of a position vector, where the position vector is defined relative to an unforced particle, and the derivative is taken with respect to an inertial frame. In the absence of a force, an unforced particle moves with constant speed along a straight line with respect to an inertial frame; consequently, any unforced particle can be used as a reference point for defining inertial acceleration.

It is possible to consider acceleration relative to an arbitrary point and with respect to a noninertial frame. An *inerter* is a device that has two physical terminals with the property that the difference between the reaction forces applied to the physical terminals is proportional to the acceleration of one of the physical terminals relative to the other, which also serves as the reference terminal [36]–[38]. Unlike the case of a mass shown in Figure 2, the reference terminal for an inerter need not have constant inertial velocity; its motion is simply the motion of one of the physical terminals.

As discussed in [37], an inerter can be realized using a ball screw or a rack and pinion. Figure S1(a) shows a ball-screw inerter where the forces applied to the physical terminals are converted to a torque, which is applied to the nut flywheel. The push-top toy in Figure S1(b) operates essentially by the same principle as the ball-screw inerter.

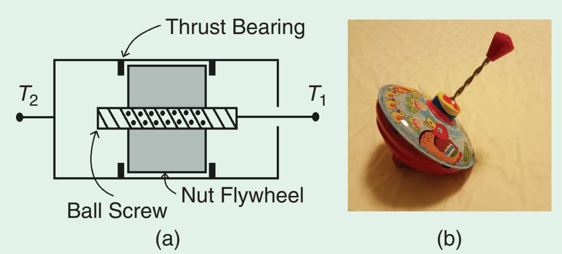

FIGURE S1 (a) A ball-screw inerter. Physical terminal T_1 is connected to a ball screw, and physical terminal T_2 is connected to the housing. The difference $\Delta f = f_2 - f_1$ between the forces applied to the physical terminals is proportional to the torque τ applied to the nut flywheel; the torque τ is proportional to the angular acceleration of the nut flywheel; and the angular acceleration of the nut flywheel is proportional to the acceleration Δa of T_2 relative to T_1. Consequently, Δf is proportional to Δa. (b) A push-top toy operates essentially by the same principle as the ball-screw inerter.

memristor and memdashpot, which are hysteretic components that relate charge to flux density and momentum to position, respectively.

This article provides a tutorial on power transmission matrices. It derives power transmission matrices for the basic mechanical components and shows how they can be combined by series and parallel connections to obtain transfer functions for specified inputs and outputs. Power transmission matrices are formulated in terms of the differential operator $\mathbf{p}$ rather than the Laplace s or its harmonic steady-state specialization $\jmath\omega$; this setting offers time-domain equations that account for arbitrary initial conditions [39].

MODELING MECHANICAL SYSTEMS USING TRANSFER FUNCTIONS

In mechanical systems, the transfer functions from force to position, force to velocity, and force to acceleration are called compliance, admittance, and accelerance, respectively. Their reciprocals are called stiffnance, impedance, and inertance, respectively. Thus, for example, impedance is the transfer function from velocity to force. Note that stiffnance, impedance, and inertance are improper transfer functions, that is, the degree of the numerator is greater than that of the denominator. This terminology is summarized in Table 1.

TABLE 1 The names of the transfer functions for structures. Admittance is the transfer function from force to velocity; impedance is its reciprocal.

	Force to Motion	Motion to Force
Position	Compliance	Stiffnance
Velocity	Admittance	Impedance
Acceleration	Accelerance	Inertance

A transmissibility is a relationship between two variables of the same type [40], [41], [42, 30 pp. 2-31–2-36]. Force-to-force, velocity-to-velocity, and position-to-position relationships constitute a force transmissibility, velocity transmissibility, and position transmissibility, respectively. As shown in [43], transmissibilities can be represented in terms of behaviors [27], [44].

To represent dynamics in the time domain, the transfer functions in this article are expressed in terms of the differential operator $\mathbf{p} \triangleq \mathrm{d}/\mathrm{d}t$ rather than the complex Laplace variable s. To clarify this distinction, note that the mass-spring system

$$m\ddot{q} + kq = f \tag{1}$$

can be written as

$$m\mathbf{p}^2 q + kq = f, \tag{2}$$

whose solution can be represented as

$$q = \frac{1}{m\mathbf{p}^2 + k} f. \tag{3}$$

Note that (3) is a representation of the time-domain solution of (1), which includes the free response due to $q(0)$ and $\dot{q}(0)$ as well as the forced response due to f. In contrast, the Laplace-domain equation

$$\hat{q}(s) = \frac{1}{ms^2 + k} \hat{f}(s) \tag{4}$$

accounts for the forced response of (1) but assumes that $q(0)$ and $\dot{q}(0)$ are zero. In fact, an additional term is needed to represent the free response in the Laplace domain. Consequently, although $G(\mathbf{p})$ in (3) and $G(s)$ in (4) have the same form, their meaning is different. Modeling based on $\mathbf{p}$ is used in behaviors as discussed in [44]. Of particular relevance to this article, a similar approach is used in [8] within the context of power transmission matrices. Further discussion about the distinction between the Laplace s and the operator $\mathbf{p}$ can be found in [39].

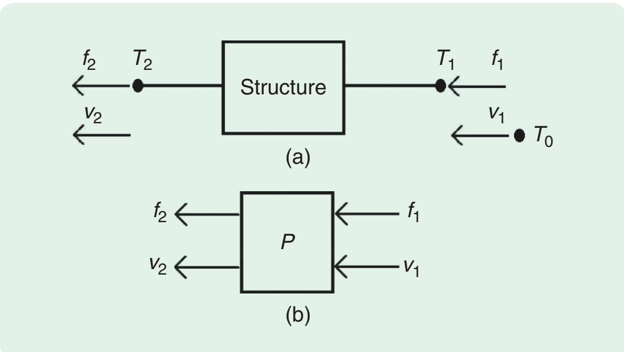

FIGURE 1 (a) A structure relating force f_1 and velocity v_1 to force f_2 and velocity v_2. The forces f_1 and f_2, which are absolute (through) variables, are applied to physical terminals T_1 and T_2, respectively. The velocities v_1 and v_2 of T_1 and T_2, respectively, are relative (across) variables that are defined relative to the reference terminal T_0. Since the axis direction is to the left, positive values of f_1 and f_2 indicate pushing or pulling to the left. Specifically, f_1 is the reaction force applied by the structure to the structure on its right, whereas f_2 is the reaction force applied to the structure by the structure on its left. By Newton's third law, these reaction forces are equal and opposite in relation to the adjacent structures. (b) The power transmission matrix P of the structure relates the force and velocity at one terminal to the force and velocity at the other terminal.

POWER TRANSMISSION MATRICES FOR MECHANICAL SYSTEMS

A power transmission matrix of a structure is a 2×2 matrix that relates the force and velocity at one physical terminal to the force and velocity at another physical terminal. Figure 1 shows a structure with force f_1 at physical terminal T_1 and force f_2 at physical terminal T_2. Force f_1 is the reaction force applied *by* the structure *to* the structure on its right, whereas f_2 is the reaction force applied *to* the structure *by* the structure on its left. This convention allows all positive forces to point to the left and avoids an excessive number of minus signs. Unlike the usual convention, the positive real axis points to the left, and positive values of f_1 and f_2 indicate that these forces are pushing to the left. The choice of axis direction has the pleasant feature that series connections constructed from right to left correspond to products of PTMs constructed from right to left. Figure 1 also shows velocities v_1 and v_2 of T_1 and T_2 relative to the reference terminal, T_0, respectively.

The *power transmission matrix P*, depicted in Figure 1, is the 2×2 matrix that relates (f_1, v_1) to (f_2, v_2) according to

$$\begin{bmatrix} f_2(t) \\ v_2(t) \end{bmatrix} = P(\mathbf{p}) \begin{bmatrix} f_1(t) \\ v_1(t) \end{bmatrix}. \tag{5}$$

Since the entries of P are functions of the differential operator $\mathbf{p}$, (5) is a time-domain equation that fully accounts for the initial conditions of all variables. Writing

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{6}$$

it follows that p_{11} is a force transmissibility (force/force), p_{12} is an impedance (force/velocity), p_{21} is an admittance (velocity/force), and p_{22} is a velocity transmissibility (velocity/velocity). The entries of a power transmission matrix consist of two transmissibilities, one impedance and one admittance. Power transmission matrices are closely related to admittance and impedance matrices. For details, see "Admittance and Impedance Matrices."

THE ELEMENTARY POWER TRANSMISSION MATRICES

In this section, the power transmission matrices are derived for the mass, inerter, spring, and dashpot; the corresponding transmission matrices are the elementary power transmission matrices. All forces and motion are assumed to occur along a line, with the positive direction in all figures taken to be toward the left. This convention leads to block diagrams that are consistent with matrix multiplication.

Consider the mass m, shown in Figure 2, whose physical terminals T_1 and T_2 are fixed attachment points on the body. The reference terminal T_0 coincides with a point with an inertial velocity that is constant and corresponds to the motion of an unforced particle [21], which may be embedded in an inertially nonrotating massive body; in analogy with circuits, an inertially nonrotating massive body can be viewed as an inertial ground. It follows from Newton's third law that

$$m a_1(t) = f_2(t) - f_1(t), \tag{7}$$

where $-f_1(t)$ is the reaction force on the structure due to the body on its right, and $a_1(t)$ is the inertial acceleration of the mass relative to T_0. It thus follows that

$$f_2(t) = f_1(t) + m\mathbf{p}v_1(t), \tag{8}$$

$$v_2(t) = v_1(t). \tag{9}$$

The elementary power transmission matrix of a mass with inertia m is thus given by

Admittance and Impedance Matrices

A power transmission matrix maps the force and velocity at one terminal to the force and velocity at another terminal. A nice feature of this formulation is that series connections can be modeled simply by forming the product of matrices of this type. Variations of these matrices map forces to velocities and velocities to forces. Specifically, by assuming that $p_{12} \neq 0$ and rearranging (5) and (6) we derive

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{p_{12}} \begin{bmatrix} -p_{11} & 1 \\ -1 & p_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{S1}$$

where the coefficient matrix is an *admittance matrix*. Assuming that $p_{21} \neq 0$, the inverse relation is given by

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = -\frac{1}{p_{21}} \begin{bmatrix} p_{22} & -1 \\ 1 & -p_{11} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{S2}$$

where the coefficient matrix is an *impedance matrix*.

As an example, the impedance matrix for a spring satisfies

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{k}{\mathbf{p}} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{S3}$$

Since this matrix is singular, it follows that the spring does not have an admittance matrix. Similarly, the admittance matrix for a mass satisfies

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{m\mathbf{p}} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{S4}$$

Since this matrix is singular, it follows that the mass does not have an impedance matrix.

Under *fI-vE*, the impedance matrix, which is the analog of the mechanical admittance, can be written as

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{S5}$$

Now, assume that $E_2 = Z_L I_2$, where Z_L is the load impedance from I_2 to E_2. The resulting input impedance, Z_{in}, from I_1 to E_1 is given by

$$Z_{\text{in}} = Z_{11} + \frac{Z_{12}Z_{21}}{Z_L - Z_{22}}. \tag{S6}$$

Alternatively, assume that $E_1 = Z_S I_1$, where Z_S is the source impedance from I_1 to E_1. The corresponding output impedance, Z_{out}, from I_2 to E_2 results from

$$Z_{\text{out}} = Z_{22} + \frac{Z_{12}Z_{21}}{Z_S - Z_{11}}. \tag{S7}$$

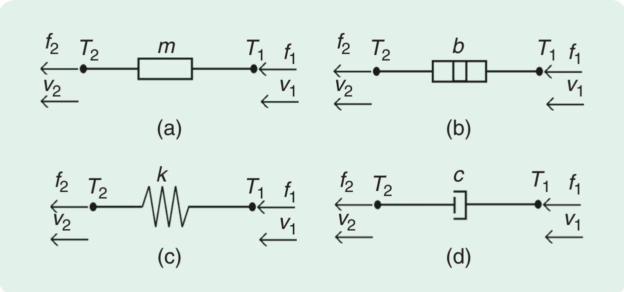

FIGURE 2 (a) A mass m with forces f_1 and f_2 acting on its physical terminals T_1 and T_2, which are rigidly attached to the body at its center of mass. The dynamics of a mass require that the reference terminal T_0 have constant inertial velocity. (b) An inerter b with forces f_1 and f_2 acting on its physical terminals T_1 and T_2. The physical terminals translate relative to each other, with velocities v_1 and v_2 relative to the reference terminal T_0. For all $t \geq 0$, $f_1(t) = f_2(t)$. (c) A spring with forces f_1 and f_2 acting on its physical terminals and velocities v_1 and v_2 relative to the reference terminal T_0. For all $t \geq 0$, $f_1(t) = f_2(t)$. (d) A dashpot with forces f_1 and f_2 acting on its physical terminals and velocities v_1 and v_2 relative to the reference terminal T_0. For all $t \geq 0$, $f_1(t) = f_2(t)$.

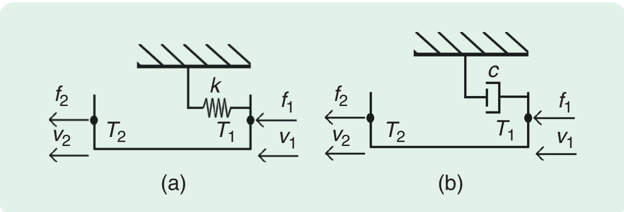

FIGURE 3 (a) A shunted spring. (b) A shunted dashpot. For both structures, T_1 and T_2 are rigidly connected to each other.

$$P_{\rm m}(m, \mathbf{p}) \triangleq \begin{bmatrix} 1 & m\mathbf{p} \\ 0 & 1 \end{bmatrix}. \tag{10}$$

A mass can be connected to other structures using one or both of its terminals. In the former case, it is a *sprung mass*, and the force on its free terminal is zero; otherwise, it is *unsprung*.

Next, consider the inerter shown in Figure 2 that has physical terminals T_1 and T_2 and reference terminal T_0. Unlike a mass, the reference terminal of an inerter need not be an inertial ground. The relative inertia b can be realized as an inerter (see "The Power Transmission Matrix of an Inerter") and satisfies

$$b(a_2(t) - a_1(t)) = f_1(t) = f_2(t), \tag{11}$$

where a_1 and a_2 are the accelerations of T_1 and T_2 relative to T_0, respectively. Therefore,

$$f_2(t) = f_1(t), \tag{12}$$
$$f_2(t) = b\mathbf{p}(v_2(t) - v_1(t)), \tag{13}$$

and thus the elementary power transmission matrix of an inerter with relative inertia b is given by

$$P_{\rm in}(b, \mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{1}{b\mathbf{p}} & 1 \end{bmatrix}. \tag{14}$$

Next, for the spring with stiffness k shown in Figure 2, note that

$$f_2(t) = f_1(t), \tag{15}$$
$$f_1(t) = k(q_2(t) - q_1(t)), \tag{16}$$

where q_1 and q_2 are the positions of T_1 and T_2 relative to T_0, respectively. It thus follows that

$$\mathbf{p}f_1(t) = k(\mathbf{p}q_2(t) - \mathbf{p}q_1(t)) = k(v_2(t) - v_1(t)), \tag{17}$$

which implies that

$$v_2(t) = \frac{\mathbf{p}}{k} f_1(t) + v_1(t). \tag{18}$$

The elementary power transmission matrix of a spring with stiffness k is thus given by

$$P_{\rm s}(k, \mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} & 1 \end{bmatrix}. \tag{19}$$

Next, for the dashpot with viscosity c shown in Figure 2, it follows that

$$f_2(t) = f_1(t), \tag{20}$$
$$f_1(t) = c(v_2(t) - v_1(t)). \tag{21}$$

Therefore,

$$v_2(t) = \frac{1}{c} f_1(t) + v_1(t), \tag{22}$$

and thus the elementary power transmission matrix of a dashpot with viscosity c is produced by

$$P_{\rm d}(c) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{1}{c} & 1 \end{bmatrix}. \tag{23}$$

Figure 3(a) shows a shunted spring k, which is a spring connected to an inertial ground. The corresponding elementary power transmission matrix is calculated by

$$P_{s,\rm sh}(k, \mathbf{p}) \triangleq \begin{bmatrix} 1 & \frac{k}{\mathbf{p}} \\ 0 & 1 \end{bmatrix}. \tag{24}$$

Similarly, Figure 3(b) shows a shunted dashpot c, which is a dashpot connected to an inertial ground. The corresponding elementary power transmission matrix is given by

$$P_{d,\rm sh}(c) \triangleq \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}. \tag{25}$$

Note that

$$\det P_m(m,\mathbf{p}) = \det P_{\text{in}}(b,\mathbf{p}) = \det P_s(k,\mathbf{p}) = \det P_d(c) = \det P_{s,\text{sh}}(k,\mathbf{p}) = \det P_{d,\text{sh}}(c) = 1. \tag{26}$$

It will be shown that all power transmission matrices formed from series and parallel connections of masses, inerters, springs, dashpots, shunted springs, and shunted dashpots share this property. Additional mechanical components are the lever and gyrator. The power transmission matrix for the lever as well as its electrical analog, the transformer, is derived in "Power Transmission Matrices for Levers and Transformers." A brief description of the gyrator is given in "What Is a Gyrator?"

RECIPROCITY

Consider the structure modeled by the power transmission matrix in Figure 1. Suppose that terminal T_2 is fixed to the reference terminal T_0. Therefore, $v_2 = 0$, and (5) becomes

$$\begin{bmatrix} f_2 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ v_1 \end{bmatrix}. \tag{27}$$

It follows that

$$f_1 = -\frac{p_{22}}{p_{21}} v_1 \tag{28}$$

and

$$f_2 = p_{11} f_1 + p_{12} v_1 \tag{29}$$

$$= \frac{p_{12}p_{21} - p_{11}p_{22}}{p_{21}} v_1. \tag{30}$$

Hence, the transfer function G_{12} from f_2 to v_1 is given by

$$G_{12} = \frac{p_{21}}{p_{12}p_{21} - p_{11}p_{22}}. \tag{31}$$

Alternatively, suppose that terminal T_1 is fixed to reference terminal T_0. In that case, $v_1 = 0$, and (5) becomes

$$\begin{bmatrix} f_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \end{bmatrix}. \tag{32}$$

Therefore,

$$v_2 = p_{21} f_1, \tag{33}$$

and the transfer function G_{21} from $-f_1$ to v_2 results from

$$G_{21} = -p_{21}, \tag{34}$$

where $-f_1$ is the force on the structure.

The structure is *reciprocal* if transfer functions G_{12} and G_{21} are equal. Therefore, the structure is reciprocal if and only if

$$\frac{p_{21}}{p_{12}p_{21} - p_{11}p_{22}} = -p_{21}. \tag{35}$$

Power Transmission Matrices for Levers and Transformers

Consider a lever of length l with endpoints e_1 and e_2, whose distances from the fulcrum are l_1 and $l_2 = l - l_1$, respectively. Letting $\omega(t)$ denote the angular velocity of the lever around the fulcrum, it follows that $v_1(t) = \omega(t) l_1$ is the velocity of e_1 along a circular arc with radius l_1, and $v_2(t) = \omega(t) l_2$ is the velocity of e_2 along a circular arc with radius l_2. Hence, $v_2(t) = (1/\lambda) v_1(t)$, where $\lambda \triangleq l_1/l_2$ is the velocity ratio. Assume that forces $-f_1$ and f_2 are applied orthogonally to the lever at e_1 and e_2, respectively, and $\omega(t)$ is constant. The total moment on the lever is zero, and thus $-f_1(t) l_1 + f_2(t) l_2 = 0$. Hence, $f_2(t) = \lambda f_1(t)$, where λ is the mechanical advantage. The power transmission matrix for the lever is given by

$$P_l(\lambda) \triangleq \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}. \tag{S8}$$

The circuit analog of the lever is the transformer, which consists of primary and secondary windings that transfer electrical energy. Let I_1 and E_1 denote the current through the primary windings and the potential drop across the primary windings, respectively; similarly, let I_2 and E_2 denote the current through the secondary windings and the potential drop across the secondary windings, respectively. Then,

$$I_2(t) = a I_1(t), \tag{S9}$$

$$E_2(t) = \frac{1}{a} E_1(t), \tag{S10}$$

where the turns ratio a is the ratio of the number of turns in the primary winding to the number of turns in the secondary winding. For $a > 1$, the transformer is step-down; for $a < 1$, it is step-up. The power transmission matrix of a transformer is produced by

$$P_t(a) \triangleq \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}. \tag{S11}$$

What Is a Gyrator?

A gyrator is an idealized electrical component whose power transmission matrix is given by

$$P_g(a) \triangleq \begin{bmatrix} 0 & \frac{1}{a} \\ a & 0 \end{bmatrix}. \tag{S12}$$

Since only the off-diagonal entries of P_g are nonzero, it follows that the gyrator converts currents to potential and vice versa. In effect, a gyrator can be used to make a capacitor emulate the signal processing ability of an inductor and vice versa. In practice, a gyrator can be approximately realized using resistors and operational amplifiers.

Hence, the structure is reciprocal if and only if

$$\det P = 1. \tag{36}$$

The following result follows from properties of series and parallel connections discussed in subsequent sections.

Proposition 1

All structures constructed from series and parallel connections of masses, inerters, springs, dashpots, and transformers are reciprocal.

Since $\det P = 1$, inverting (5) yields

$$\begin{bmatrix} f_1(t) \\ v_1(t) \end{bmatrix} = P(\mathbf{p})^{-1} \begin{bmatrix} f_2(t) \\ v_2(t) \end{bmatrix}, \tag{37}$$

where

$$P^{-1} = \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}. \tag{38}$$

Consequently, p_{11} and p_{22} can be viewed as force and velocity transmissibilities.

SYMMETRY

Let P be the power transmission matrix of the structure $\mathcal{S}$ with inputs (f_1, v_1) and outputs (f_2, v_2) diagrammed in Figure 1 and modeled by (5). Now, let $\mathcal{S}_{\rm rev}$ be the structure, with T_1 relabeled as T_2 and vice versa. This relabeling is equivalent to reversing the direction of the positive axis, which replaces v_1 and v_2 with $-v_2$ and $-v_1$, respectively. Furthermore, f_1 and f_2 become $-f_2$ and $-f_1$, respectively. The interpretation of these forces is the opposite of the convention for defining forces along the positive axis; under that convention, $-f_2$ and $-f_1$ are equivalent to f_2 and f_1, respectively. Therefore, the power transmission matrix $P_{\rm rev}$ of $\mathcal{S}_{\rm rev}$ satisfies

$$\begin{bmatrix} f_1 \\ -v_1 \end{bmatrix} = P_{\rm rev} \begin{bmatrix} f_2 \\ -v_2 \end{bmatrix}. \tag{39}$$

$P_{\rm rev}$ is the reverse power transmission matrix of $\mathcal{S}$, and the structure is symmetric if $P_{\rm rev} = P$.

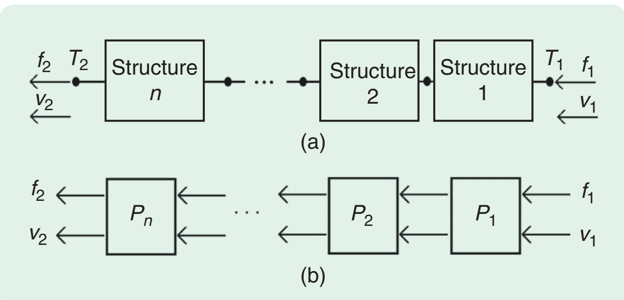

FIGURE 4 A series connection of n structures. (a) The structures are connected in series by rigidly joining their physical terminals, where T_1 and T_2 are the physical terminals of the first and last structure, respectively. (b) A block diagram of the series connection in terms of transmission matrices.

Next, multiplying (39) by $D \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it follows that

$$\begin{bmatrix} f_1 \\ v_1 \end{bmatrix} = DP_{\rm rev} D \begin{bmatrix} f_2 \\ v_2 \end{bmatrix}. \tag{40}$$

On the other hand, inverting P yields

$$\begin{bmatrix} f_1 \\ v_1 \end{bmatrix} = P^{-1} \begin{bmatrix} f_2 \\ v_2 \end{bmatrix}. \tag{41}$$

Hence, $DP_{\rm rev}D = P^{-1}$. The structure is symmetric if and only if $P^{-1} = DPD$ and the diagonal entries of P are identical. It follows from the symmetry condition $P^{-1} = DPD$ that $|\det P| = 1$. By reciprocity, the stronger condition $\det P = 1$ holds.

Intuitively, the mass, inerter, spring, and dashpot are symmetric structures, since the effect of applying a force and velocity at one terminal on the other terminal does not change if the terminals are relabeled. In fact, $P_m^{-1}(m, \mathbf{p}) = DP_m(m, \mathbf{p})D$, $P_{\rm in}^{-1}(b, \mathbf{p}) = DP_{\rm in}(b, \mathbf{p})D$, $P_s(k, \mathbf{p})^{-1} = DP_s(k, \mathbf{p})D$, and $P_d(c, \mathbf{p})^{-1} = DP_d(c, \mathbf{p})D$. However, many structures are not symmetric, as shown by Example 1.

POWER TRANSMISSION MATRICES FOR SERIES CONNECTIONS

The series connection of a pair of structures requires that the neighboring physical terminals be rigidly joined. Because of the convention that left-side arrows point left and right-side arrows point right, it follows from Newton's third law that adjacent forces at the common physical terminal must be equal, that is, have the same sign and magnitude. A single common reference terminal is assumed for the entire structure. The reference terminal must have constant inertial velocity in the case where at least one structure in the series connection is a mass; otherwise, it may be arbitrary.

Consider the series connection of the structures shown in Figure 4, where, for all $i = 1, \ldots, n-1$,

$$f_{i,2} = f_{i+1,1}, \tag{42}$$
$$v_{i,2} = v_{i+1,1}, \tag{43}$$

where $f_{i,1}$ and $f_{i,2}$ and $v_{i,1}$ and $v_{i,2}$ are the forces and velocities of the ith structure in the figure. For all $i = 1, \ldots, n$, let P_i be the power transmission matrix of the ith structure, that is,

$$\begin{bmatrix} f_{i,2} \\ v_{i,2} \end{bmatrix} = P_i \begin{bmatrix} f_{i,1} \\ v_{i,1} \end{bmatrix}, \tag{44}$$

where

$$P_i \triangleq \begin{bmatrix} p_{i,11} & p_{i,12} \\ p_{i,21} & p_{i,22} \end{bmatrix}. \tag{45}$$

Note that, for convenience, the arguments of P_i are omitted. Henceforth, the time argument will also be excluded.

To find the equivalent transmission matrix that relates (f_1, v_1) to (f_2, v_2) in Figure 4, note that

$$\begin{bmatrix} f_2 \\ v_2 \end{bmatrix} = P_n \begin{bmatrix} f_{n,1} \\ v_{n,1} \end{bmatrix} = P_n P_{n-1} \begin{bmatrix} f_{n-1,1} \\ v_{n-1,1} \end{bmatrix} = P_n P_{n-1} \cdots P_1 \begin{bmatrix} f_1 \\ v_1 \end{bmatrix}. \quad (46)$$

Therefore, the power transmission matrix that results from connecting $P_1, \ldots, P_n$ in series is given by

$$P_{\mathrm{ser}} \triangleq P_n P_{n-1} \cdots P_1. \quad (47)$$

Note that the ordering of the factors in (47) depends on the left-to-right ordering, from (f_1, v_1) to (f_2, v_2), of the structures with the power transmission matrices $P_1, \ldots, Pn$, respectively, depicted in Figure 4. Since transmission matrices may not commute, the product (47) depends on the physical ordering of the structures in the series connection. Observe that, if P_{ser} is the product of elementary power transmission matrices, det $P_{\mathrm{ser}} = 1$.

Example 1: Series Connections

Consider the series connection of two springs with stiffnesses k_1 and k_2 illustrated in Figure 5(a). The power transmission matrix of this series connection is given by

$$\begin{aligned} P_{\mathrm{ser}}(\mathbf{p}) &= P_s(k_2, \mathbf{p}) P_s(k_1, \mathbf{p}) \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\mathbf{p} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1}} & 1 \end{bmatrix}, \end{aligned} \quad (48)$$

which shows that the equivalent stiffness is $((1/k_1) + (1/k_2))^{-1}$. The equivalent viscosity of the series connection of two dashpots with viscosities c_1 and c_2 shown in Figure 5(b) is $((1/c_1) + (1/c_2))^{-1}$. The power transmission matrix of the series connection of a dashpot with viscosity c and a spring with stiffness k in Figure 5(c) results from

$$\begin{aligned} P_{\mathrm{ser}}(\mathbf{p}) &= P_s(k, \mathbf{p}) P_d(c, \mathbf{p}) \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{c} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} + \frac{1}{c} & 1 \end{bmatrix}, \end{aligned} \quad (49)$$

which shows that the admittance of a spring and dashpot in series is given by $(\mathbf{p}/k_1) + (1/c)$. Note that these three series connections are symmetric structures. The power transmission matrix of the series connection of a spring with stiffness k and a mass with inertia m in Figure 5(d) is produced by

$$\begin{aligned} P_{\mathrm{ser}}(\mathbf{p}) &= P_s(k, \mathbf{p}) P_m(m, \mathbf{p}) \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} & 1 \end{bmatrix} \begin{bmatrix} 1 & m\mathbf{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m\mathbf{p} \\ \frac{\mathbf{p}}{k} & \frac{m\mathbf{p}^2}{k} + 1 \end{bmatrix}, \end{aligned} \quad (50)$$

which is not symmetric. Finally, the power transmission matrix of the series connection of a spring with stiffness k and an inerter with relative inertia b in Figure 5(e) is calculated by

$$\begin{aligned} P_{\mathrm{ser}}(\mathbf{p}) &= P_s(k, \mathbf{p}) P_{\mathrm{in}}(b, \mathbf{p}) \\ &= \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{b\mathbf{p}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} + \frac{1}{b\mathbf{p}} & 1 \end{bmatrix}, \end{aligned} \quad (51)$$

which is symmetric.

POWER TRANSMISSION MATRICES FOR PARALLEL CONNECTIONS

The parallel connection of two structures requires that the physical terminals on each side of the structures be rigidly joined together. Consequently, on each side of a parallel connection, the velocities of the physical terminals relative to the reference terminal are equal.

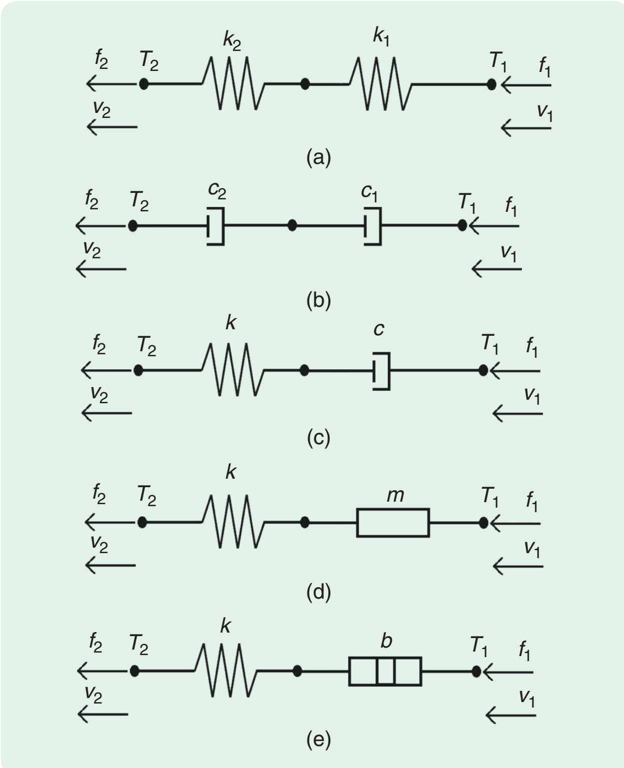

FIGURE 5 (a) A series connection of two springs with stiffnesses k_1 and k_2. (b) A series connection of two dashpots with viscosities c_1 and c_2. (c) A series connection of a dashpot with viscosity c and a spring with stiffness k. (d) A series connection of a mass with inertia m and a spring with stiffness k. (e) A series connection of an inerter with relative inertia b and a spring with stiffness k.

Consider the parallel connection of the structures illustrated in Figure 6, where, for all $i, j \in \{1, \ldots, n\}$,

$$v_{i,1} = v_{j,1}, \tag{52}$$
$$v_{i,2} = v_{j,2}, \tag{53}$$

and

$$f_1 = \sum_{i=1}^{n} f_{i,1}, \tag{54}$$

$$f_2 = \sum_{i=1}^{n} f_{i,2}, \tag{55}$$

where $f_{i,1}$ and $f_{i,2}$ are the forces at the common physical terminals, and $v_{i,1}$ and $v_{i,2}$ are the velocities of the common physical terminals. Let

$$P_i \triangleq \begin{bmatrix} p_{i,11} & p_{i,12} \\ p_{i,21} & p_{i,22} \end{bmatrix} \tag{56}$$

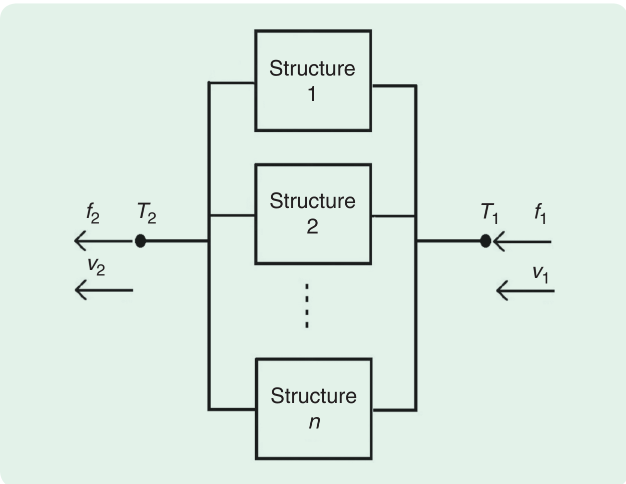

FIGURE 6 The parallel connection of n structures. The structures are connected in parallel by rigidly joining their right physical terminals to the physical terminal T_1 and their left physical terminals to the physical terminal T_2.

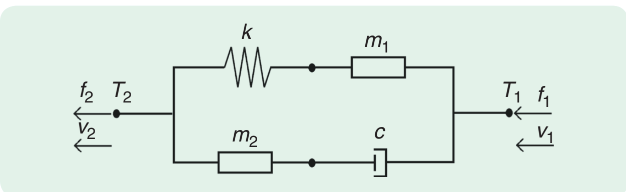

FIGURE 7 The series connection of a mass and spring is linked in parallel with the series connection of a dashpot and mass.

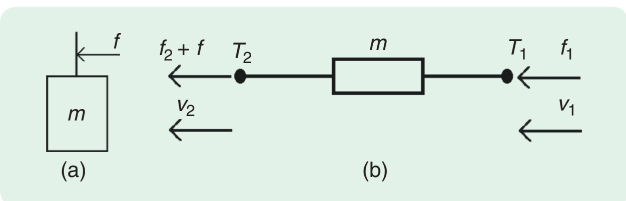

FIGURE 8 (a) A mass with an external force, f, acting on it. (b) The external force f is applied to the terminal to the left of m.

be the power transmission matrix that represents the ith structure in the parallel connection in Figure 6.

If two masses with inertias m_1 and m_2 are connected in parallel, the resulting structure is a mass with inertia $m_1 + m_2$. If a mass is connected in parallel with a spring, it constrains the motion of the physical terminals, and the spring cannot be compressed. The same situation occurs where a mass is connected in parallel with a dashpot. The instance where one of the structures in parallel is a mass is not considered. However, each structure in a parallel connection may be a series connection of masses and at least one inerter, spring, or dashpot in series.

As shown in "The Derivation of P_{par}," the power transmission matrix P_{par} that relates (f_1, v_1) to (f_2, v_2) in Figure 6 is calculated by

$$P_{\text{par}} = \frac{1}{\beta}\begin{bmatrix} \alpha & \alpha\gamma - \beta^2 \\ 1 & \gamma \end{bmatrix}, \tag{57}$$

where

$$\alpha \triangleq \sum_{i=1}^{n} \frac{p_{i,11}}{p_{i,21}}, \quad \beta \triangleq \sum_{i=1}^{n} \frac{1}{p_{i,21}}, \quad \gamma \triangleq \sum_{i=1}^{n} \frac{p_{i,22}}{p_{i,21}}. \tag{58}$$

Note that division by zero would occur if one of the structures were a mass; however, as discussed above, this case is not allowed. The expressions of (58) are given in [45], [46], and [7, pp. 10–36]. Related expressions are given in [13, pp. 342, 343] and [47].

Example 2: Parallel Connection

Consider the parallel connection of two structures, where one structure is the series connection of a mass with inertia m_1 and a spring with stiffness k, and the other one is the series connection of a mass with inertia m_2 and a dashpot with viscosity c, shown in Figure 7. The power transmission matrix of the series connection of the mass and the spring is given by

$$P_1(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ \frac{\mathbf{p}}{k} & 1 \end{bmatrix}\begin{bmatrix} 1 & m_1\mathbf{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m_1\mathbf{p} \\ \frac{\mathbf{p}}{k} & \frac{m_1\mathbf{p}^2}{k} + 1 \end{bmatrix}, \tag{59}$$

while the power transmission matrix of the series connection of the dashpot and the mass is given by

$$P_1(\mathbf{p}) = \begin{bmatrix} 1 & m_2\mathbf{p} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{c} & 1 \end{bmatrix} = \begin{bmatrix} \frac{m_2\mathbf{p}}{c} + 1 & m_2\mathbf{p} \\ \frac{1}{c} & 1 \end{bmatrix}. \tag{60}$$

Applying (57) and (58) to (59) and (60) yields

$$P_{\text{par}}(\mathbf{p}) = \frac{1}{c\mathbf{p} + k}\begin{bmatrix} m_2\mathbf{p}^2 + c\mathbf{p} + k & m_1 m_2 \mathbf{p}^3 + (m_1 + m_2)(c\mathbf{p}^2 + k\mathbf{p}) \\ \mathbf{p} & m_1\mathbf{p}^2 + c\mathbf{p} + k \end{bmatrix}, \tag{61}$$

which is the power transmission matrix of the parallel connection in Figure 7. It can be seen that $\det P_{\text{par}} = 1$; however, the structure is not symmetric. ◇

MODELING EXTERNAL FORCES

Consider mass m pictured in Figure 8, where f is an external force acting on it. Using Newton's second law, it follows from Figure 8 that

$$m\mathbf{p}v_1(t) = f_2(t) + f(t) - f_1(t). \tag{62}$$

Since $v_2 = v_1$, it follows that (f_1, v_1) and (f_2, v_2) are related by

$$\begin{bmatrix} f_2(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 1 & m\mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ v_1(t) \end{bmatrix} + \begin{bmatrix} -f(t) \\ 0 \end{bmatrix} \tag{63}$$

or, equivalently,

$$\begin{bmatrix} f_2(t) + f(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 1 & m\mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ v_1(t) \end{bmatrix}. \tag{64}$$

MODELING STRUCTURES WITH BOUNDARY CONDITIONS

Example 3: The Single-Degree-of-Freedom Structure

Consider the structure depicted in Figure 9(a), where external force f is applied to the mass. The inerter, spring, and dashpot are connected in parallel, and the mass is connected in series with the inerter, spring, and dashpot. Figure 9(b) shows the physical terminals of the structure. It

The Derivation of P_{par}

We consider the parallel connection of $n = 2$ power transmission matrices; the case of n power transmission matrices follows by induction. First, define

$$P_1 = \begin{bmatrix} p_{1,11} & p_{1,12} \\ p_{1,21} & p_{1,22} \end{bmatrix} \tag{S13}$$

$$P_2 = \begin{bmatrix} p_{2,11} & p_{2,12} \\ p_{2,21} & p_{2,22} \end{bmatrix} \tag{S14}$$

so that

$$\begin{bmatrix} f_{12} \\ v_2 \end{bmatrix} = P_1 \begin{bmatrix} f_{11} \\ v_1 \end{bmatrix}, \quad \begin{bmatrix} f_{22} \\ v_2 \end{bmatrix} = P_2 \begin{bmatrix} f_{21} \\ v_1 \end{bmatrix}, \tag{S15}$$

where v_2 is the common velocity of the physical output terminals of P_1 and P_2. It follows that

$$v_2 = p_{1,21} f_{11} + p_{1,22} v_1, \tag{S16}$$

$$v_2 = p_{2,21} f_{21} + p_{2,22} v_1, \tag{S17}$$

which imply that

$$p_{1,11} f_{11} = \frac{p_{1,11}}{p_{1,21}} v_2 - \frac{p_{1,11} p_{1,22}}{p_{1,21}} v_1, \tag{S18}$$

$$p_{2,11} f_{21} = \frac{p_{2,11}}{p_{2,21}} v_2 - \frac{p_{2,11} p_{2,22}}{p_{2,21}} v_1. \tag{S19}$$

Adding (S18) and (S19) yields

$$p_{1,11} f_{11} + p_{2,11} f_{21} = \alpha v_2 - \delta v_1, \tag{S20}$$

where

$$\delta \triangleq \frac{p_{1,11} p_{1,22}}{p_{1,21}} + \frac{p_{2,11} p_{2,22}}{p_{2,21}}. \tag{S21}$$

Next, dividing (S16) by $P_{1,21}$ and (S17) by $P_{2,21}$, adding the resulting equations, and dividing by β yields

$$v_2 = \frac{1}{\beta}(f_{11} + f_{21}) + \frac{\gamma}{\beta} v_1. \tag{S22}$$

Now, substituting (S22) into (S20) results in

$$p_{1,11} f_{11} + p_{2,11} f_{21} = \frac{\alpha}{\beta}(f_{11} + f_{21}) + \left(\frac{\alpha\gamma}{\beta} - \delta\right) v_1. \tag{S23}$$

Next, summing

$$f_{12} = p_{1,11} f_{11} + p_{1,12} v_1 \tag{S24}$$

$$f_{22} = p_{2,11} f_{21} + p_{2,12} v_1 \tag{S25}$$

yields

$$f_{12} + f_{22} = p_{1,11} f_{11} + p_{2,11} f_{21} + (p_{1,12} + p_{2,12}) v_1. \tag{S26}$$

Substituting (S23) into (S26) gives

$$\begin{aligned} f_{12} + f_{22} &= \frac{\alpha}{\beta}(f_{11} + f_{21}) + \left(\frac{\alpha\gamma}{\beta} + p_{1,12} + p_{2,12} - \delta\right) v_1 \\ &= \frac{1}{\beta}[\alpha(f_{11} + f_{21}) + (\alpha\gamma + \beta[p_{1,12} + p_{2,12} - \delta]) v_1] \\ &= \frac{1}{\beta}\left[\alpha(f_{11} + f_{21}) + \left(\alpha\gamma - \beta\left[\frac{\det P_1}{p_{1,21}} + \frac{\det P_2}{p_{2,21}}\right]\right) v_1\right]. \end{aligned} \tag{S27}$$

Now consider the case where P_1 and P_2 represent series connections of masses, springs, and dashpots. It follows that $\det P_1 = \det P_2 = 1$, and thus (S27) implies that

$$f_{12} + f_{22} = \frac{1}{\beta}[\alpha(f_{11} + f_{21}) + (\alpha\gamma - \beta^2) v_1]. \tag{S28}$$

Combining (S22) and (S28) yields (57). This proves (57) in the case where P_1 and P_2 are each the power transmission matrices of the series connections of masses, springs, and dashpots. Note that $\det P_{\text{par}} = 1$.

Consider the case where P_1 and P_2 are each either the power transmission matrices of the series connections of masses, springs, and dashpots (as in the previous instance) or the power transmission matrices of the series connections of parallel connections of masses, springs, and dashpots. In this case, $\det P_1 = \det P_2 = 1$, and again (57) holds. Continuing in a similar fashion, it follows that, for all structures constructed from series and parallel connections of masses, springs, and dashpots, (57) holds. ◇

follows from (57) that the power transmission matrix P of the parallel inerter-spring-dashpot connection is given by

$$P(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ \dfrac{\mathbf{p}}{b\mathbf{p}^2 + c\mathbf{p} + k} & 1 \end{bmatrix}. \tag{65}$$

The wall is assumed to be an inertially nonrotating massive body, and it serves as physical terminal T_1 and reference terminal T_0. Let f_1 represent the force applied to the wall by the spring and dashpot, and since T_0 is colocated with T_1, it follows that $v_1 = 0$. Viewing the left edge of the mass as physical terminal T_2, let v_2 represent the velocity of the mass. Since no force is applied to the left edge of the mass, it is clear that $f_2 = 0$. Using the boundary conditions $v_1 = 0$ and $f_2 = 0$, it follows from (64) that

$$\begin{bmatrix} f(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 1 & m\mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{\mathbf{p}}{b\mathbf{p}^2 + c\mathbf{p} + k} & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(t) + \dfrac{m\mathbf{p}^2}{b\mathbf{p}^2 + c\mathbf{p} + k} f_1(t) \\ \dfrac{\mathbf{p}}{b\mathbf{p}^2 + c\mathbf{p} + k} f_1(t) \end{bmatrix}. \tag{66}$$

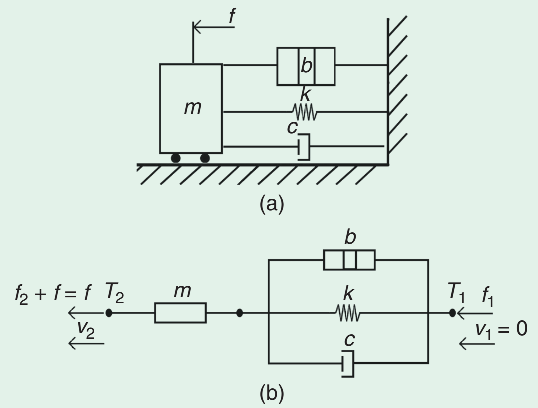

FIGURE 9 (a) An inerter, spring, and dashpot are connected in parallel; this substructure is connected in series with a mass. The external force f is applied to the mass, which is sprung. (b) The physical terminal T_1 is attached to the wall and thus has zero velocity. The physical terminal T_2 is attached to the left side of the mass and is free.

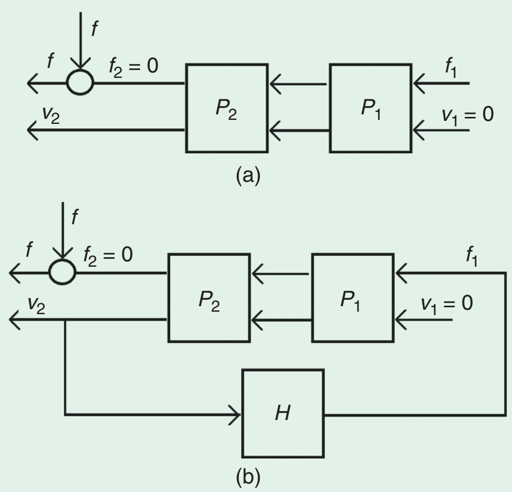

FIGURE 10 (a) A block diagram of the single-degree-of-freedom structure in Figure 9 represented as the product of two elementary power transmission matrices. The external force f is added to f_2, which is zero since the mass is sprung. The goal is to determine the transfer function from the external force f to the velocity v_2 of m_2. (b) A modification of the block diagram in (a). The transfer function H given by (70) specifies the force f_1 on the wall in terms of v_2. By relating v_2 to f_1, H closes a feedback loop.

It thus follows that

$$f(t) = \frac{(m+b)\mathbf{p}^2 + c\mathbf{p} + k}{b\mathbf{p}^2 + c\mathbf{p} + k} f_1(t), \tag{67}$$

$$v_2(t) = \frac{\mathbf{p}}{b\mathbf{p}^2 + c\mathbf{p} + k} f_1(t). \tag{68}$$

Combining (67) and (68) yields

$$v_2(t) = \frac{\mathbf{p}}{(m+b)\mathbf{p}^2 + c\mathbf{p} + k} f(t). \tag{69}$$

Alternatively, solving (68) for f_1 gives

$$f_1(t) = \frac{b\mathbf{p}^2 + c\mathbf{p} + k}{\mathbf{p}} v_2(t), \tag{70}$$

which expresses the unknown force f_1 applied to the wall by the structure in terms of v_2. Substituting f_1 given by (70) into (67) and solving for v_2 yields (69). Although this different derivation of (69) involves more steps, it utilizes (68), which can be viewed as a feedback connection, as illustrated in Figure 10 [where H denotes the transfer function in (68)]. This feedback connection specifies the unknown reaction force f_1 applied by the wall in terms of the velocity, v_2, of the mass. ◇

Example 4: The Two-Degrees-of-Freedom Structure

Consider the structure pictured in Figure 11(a), where the external force f is applied to the first mass. Note that the springs and masses are connected in series. Figure 11(b) shows the physical terminals of the structure. Using the boundary conditions $v_1 = 0$ and $f_2 = 0$, it follows from (63) that

$$\begin{bmatrix} 0 \\ v_2(t) \end{bmatrix} = P_m(m_2, \mathbf{p}) P_s(k_2, \mathbf{p}) \left(P_m(m_1, \mathbf{p}) P_s(k_1, \mathbf{p}) \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} -f(t) \\ 0 \end{bmatrix} \right) \tag{71}$$

$$= \begin{bmatrix} \dfrac{1}{k_1 k_2}[(m_1 m_2 \mathbf{p}^4 + (k_2 m_1 + k_2 m_2 + k_1 m_2)\mathbf{p}^2 + k_1 k_2) f_1(t) \\ -(k_1 m_2 \mathbf{p}^2 + k_1 k_2) f(t)] \\ \dfrac{1}{k_1 k_2}[(m_1 \mathbf{p}^3 + k_1 \mathbf{p} + k_2 \mathbf{p}) f_1(t) - k_1 \mathbf{p} f(t)] \end{bmatrix}, \tag{72}$$

as illustrated in Figure 12(a). Therefore,

$$0 = [m_1 m_2 \mathbf{p}^4 + (k_1 m_2 + k_2 m_1 + k_2 m_2)\mathbf{p}^2 + k_1 k_2] f_1(t) - k_1(m_2 \mathbf{p}^2 + k_2) f(t), \tag{73}$$

$$v_2(t) = \frac{1}{k_1 k_2}[(m_1 \mathbf{p}^3 + k_1 \mathbf{p} + k_2 \mathbf{p}) f_1(t) - k_1 \mathbf{p} f(t)]. \tag{74}$$

Solving (73) for $f_1(t)$ yields

$$f_1(t) = \frac{k_1(m_2 \mathbf{p}^2 + k_2)}{m_1 m_2 \mathbf{p}^4 + (k_1 m_2 + k_2 m_1 + k_2 m_2)\mathbf{p}^2 + k_1 k_2} f(t). \tag{75}$$

Finally, substituting (75) into (74) gives

$$v_2(t) = \frac{k_2 \mathbf{p}}{(m_1 \mathbf{p}^2 + k_1)(m_2 \mathbf{p}^2 + k_2) + k_2 m_2 \mathbf{p}^2} f(t). \tag{76}$$

Alternatively, solving (73) for $f(t)$ produces

$$f(t) = \frac{m_1 m_2 \mathbf{p}^4 + (k_1 m_2 + k_2 m_1 + k_2 m_2)\mathbf{p}^2 + k_1 k_2}{k_1(m_2 \mathbf{p}^2 + k_2)} f_1(t). \tag{77}$$

Next, substituting $f(t)$ given by (77) into (74) results in

$$v_2(t) = \frac{k_2 \mathbf{p}(m_1 m_2 \mathbf{p}^4 + (k_1 m_2 + k_2 m_1 + k_2 m_2)\mathbf{p}^2 + k_1 k_2)}{k_1(m_2 \mathbf{p}^2 + k_2)[(m_1 \mathbf{p}^2 + k_1)(m_2 \mathbf{p}^2 + k_2) + k_2 m_2 \mathbf{p}^2]} f_1(t). \tag{78}$$

Therefore,

$$f_1(t) = \frac{k_1(m_2 \mathbf{p}^2 + k_2)[(m_1 \mathbf{p}^2 + k_1)(m_2 \mathbf{p}^2 + k_2) + k_2 m_2 \mathbf{p}^2]}{k_2 \mathbf{p}(m_1 m_2 \mathbf{p}^4 + (k_1 m_2 + k_2 m_1 + k_2 m_2)\mathbf{p}^2 + k_1 k_2)} v_2(t), \tag{79}$$

which expresses the unknown force f_1 applied by the wall in terms of v_2. Substituting $f_1(t)$ given by (79) into (74) and solving for $v_2(t)$ yields (76). While this derivation of (76) is more tedious, it utilizes (79), which can be viewed as a feedback connection, as depicted in Figure 12(b), where H denotes the transfer function in (79). This feedback connection specifies the unknown reaction force f_1 applied by the wall. ◇

ENERGY TRANSMISSION MATRICES FOR STRUCTURES

An energy transmission matrix, E, of a structure with physical terminals T_1 and T_2 is a 2×2 matrix that describes the relationship between the forces f_1 and f_2 and the positions q_1 and q_2 of T_1 and T_2, as shown in Figure 13(a). As depicted in Figure 13(b), E satisfies

$$\begin{bmatrix} f_2(t) \\ q_2(t) \end{bmatrix} = E(\mathbf{p}) \begin{bmatrix} f_1(t) \\ q_1(t) \end{bmatrix}, \tag{80}$$

where

$$E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}. \tag{81}$$

Note that e_{11} is a force transmissibility, e_{12} is a stiffnance, e_{21} is a compliance, and e_{22} is a position transmissibility. The entries of a transmission matrix consist of two transmissibilities, one stiffnance and one compliance.

Relating energy transmission matrices to power transmission matrices using $v_1 = \mathbf{p} q_1$ and $v_2 = \mathbf{p} q_2$ yields

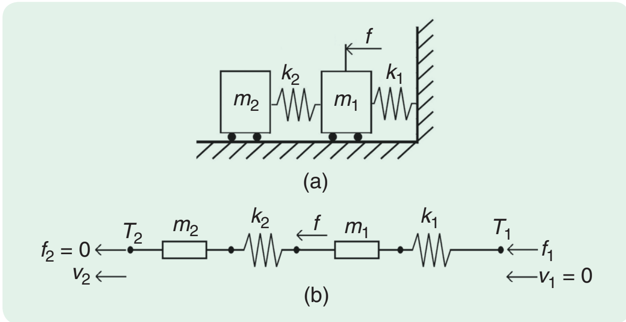

FIGURE 11 (a) A structure with two masses and two springs. (b) The springs k_1 and k_2 and masses m_1 and m_2 are connected in series, and the external force f is applied to the terminal to the left of m_1.

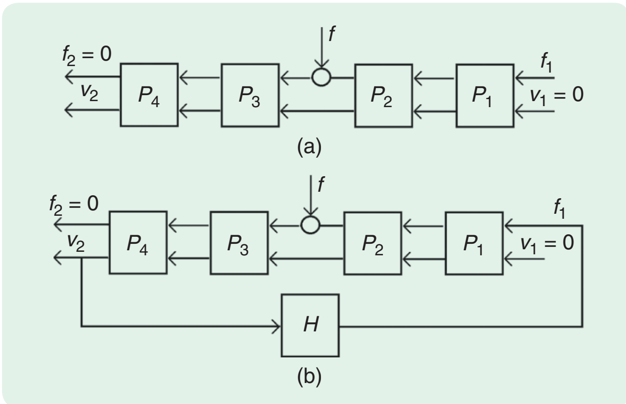

FIGURE 12 (a) A block diagram of the two-degrees-of-freedom structure in Figure 11 represented as the product of four elementary power transmission matrices. The external force f is added to the output of m_2. The goal is to determine the transfer function from the external force f to the velocity v_2 of m_2. (b) A modification of the block diagram in (a). The transfer function H given by (79) specifies the force f_1 on the wall in terms of v_2. By relating v_2 to f_1, H closes a feedback loop.

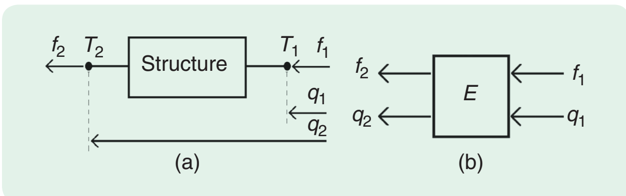

FIGURE 13 (a) A structure relating force f_1 and position q_1 to force f_2 and position q_2. The forces f_1 and f_2, which are absolute (through) variables, are applied to the physical terminals T_1 and T_2, respectively. The positions q_1 and q_2 of T_1 and T_2, respectively, are relative (across) variables defined relative to the reference terminal, T_0. (b) The input–output representation of (5) showing the input signals f_1 and q_1 and output signals f_2 and q_2. The energy transmission matrix $E(\mathbf{p})$ of the structure represents the relationship between the force and position of one physical terminal to the force and position of another physical terminal.

$$E=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{\mathbf{p}}\end{bmatrix}P\begin{bmatrix}1 & 0\\ 0 & \mathbf{p}\end{bmatrix}=\begin{bmatrix}p_{11} & \mathbf{p}p_{12}\\ \frac{p_{21}}{\mathbf{p}} & p_{22}\end{bmatrix}. \tag{82}$$

The elementary energy transmission matrices for the mass, inerter, spring, and dashpot are given by

$$E_m(m,\mathbf{p}) \triangleq \begin{bmatrix}1 & m\mathbf{p}^2\\ 0 & 1\end{bmatrix}, \tag{83}$$

$$E_{\text{in}}(b,\mathbf{p}) \triangleq \begin{bmatrix}1 & 0\\ \frac{1}{b\mathbf{p}^2} & 1\end{bmatrix}, \tag{84}$$

$$E_s(k,\mathbf{p}) \triangleq \begin{bmatrix}1 & 0\\ \frac{1}{k} & 1\end{bmatrix}, \tag{85}$$

$$E_d(c,\mathbf{p}) \triangleq \begin{bmatrix}1 & 0\\ \frac{1}{c\mathbf{p}} & 1\end{bmatrix}. \tag{86}$$

As in the case of the elementary power transmission matrices,

$$\begin{aligned}\det E_m(m,\mathbf{p}) &= \det E_{in}(b,\mathbf{p})\\ &= \det E_s(k,\mathbf{p}) = \det E_d(c,\mathbf{p}) = 1.\end{aligned} \tag{87}$$

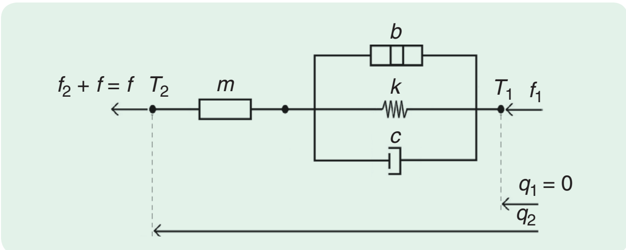

FIGURE 14 The physical terminal T_1 is attached to the wall and thus has zero position. The physical terminal T_2 is attached to the left side of the mass and is free.

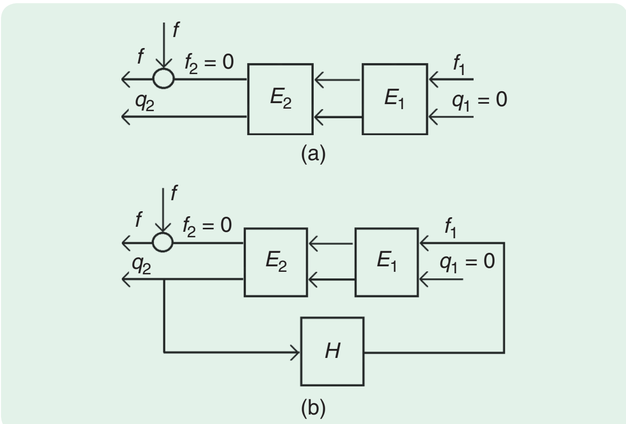

FIGURE 15 (a) A block diagram of the single-degree-of-freedom structure in Figure 9 represented as the product of two elementary energy transmission matrices. The external force f is added to f_2, which is zero since the mass is sprung. The goal is to determine the transfer function from the external force f to the position q_2 of m_2. (b) A modification of the block diagram in (a). The transfer function H given by (94) specifies the force f_1 on the wall in terms of q_2. By relating q_2 to f_1, H closes a feedback loop.

The energy transmission matrices for series and parallel connections satisfy the rules given for power transmission matrices.

Example 5: Energy Analysis of the Single-Degree-of-Freedom Structure

Consider the structure in Figure 9(a), where external force f is applied to the mass. The inerter, spring, and dashpot are connected in parallel, and the mass is connected in series with the inerter, spring, and dashpot. Figure 14 shows the physical terminals of the structure. It follows from (57) that the energy transmission matrix E of the parallel inerter-spring-dashpot connection is given by

$$E(\mathbf{p}) = \begin{bmatrix}1 & 0\\ \frac{1}{b\mathbf{p}^2+c\mathbf{p}+k} & 1\end{bmatrix}. \tag{88}$$

Let f_1 represent the force applied to the wall by the spring and dashpot, and since T_0 is colocated with T_1, it follows that $q_1 = 0$. Viewing the left edge of the mass as physical terminal T_2, let q_2 represent the position of the mass from the reference terminal, T_0. Because no force is applied to the left edge of the mass, it follows that $f_2 = 0$. Using the boundary conditions $q_1 = 0$ and $f_2 = 0$, it follows from (64) that

$$\begin{aligned}\begin{bmatrix}f(t)\\ q_2(t)\end{bmatrix} &= \begin{bmatrix}1 & m\mathbf{p}^2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{1}{b\mathbf{p}^2+c\mathbf{p}+k} & 1\end{bmatrix}\begin{bmatrix}f_1(t)\\ 0\end{bmatrix}\\ &= \begin{bmatrix}f_1(t)+\frac{m\mathbf{p}^2}{b\mathbf{p}^2+c\mathbf{p}+k}f_1(t)\\ \frac{1}{b\mathbf{p}^2+c\mathbf{p}+k}f_1(t)\end{bmatrix}.\end{aligned} \tag{89}$$

It thus follows that

$$f(t) = \frac{(m+b)\mathbf{p}^2+c\mathbf{p}+k}{b\mathbf{p}^2+c\mathbf{p}+k}f_1(t), \tag{90}$$

$$q_2(t) = \frac{1}{b\mathbf{p}^2+c\mathbf{p}+k}f_1(t). \tag{91}$$

Combining (90) and (91) results in

$$q_2(t) = \frac{1}{(m+b)\mathbf{p}^2+c\mathbf{p}+k}f(t). \tag{92}$$

Finally, as in the case of power transmission matrices, the feedback connection is shown in Figure 15, where H denotes the transfer function that satisfies

$$f_1(t) = H(\mathbf{p})q_2(t), \tag{93}$$

that is,

$$H(\mathbf{p}) = b\mathbf{p}^2+c\mathbf{p}+k. \tag{94}$$

◇

POWER TRANSMISSION MATRICES FOR ELECTRICAL SYSTEMS

In electrical systems, power is the product of potential E and current I. The unit of potential is the volt, where 1 V = 1 joule/coulomb (J/C). The unit of current is the ampere, where 1 A = 1 C/s. Current is the derivative of charge Q, that is, $I = \dot{Q}$. The product EI has the dimensions of power, whose unit is the watt, where 1 W = (1 V)(1 A) = (1 J/C)(1 C/s) = 1 J/s.

As with structures, power transmission matrices can be defined for circuits. This can be done by expressing an analogy between mechanical systems and electrical systems and recasting the results obtained for mechanical systems. Taking advantage of the fact that the product of current and potential has the dimensions of power, this can be done by 1) associating force (f) with potential (E) and velocity (v) with current (I) (the fE-vI analogy) or 2) associating force (f) with current (I), and velocity (v) with potential (E) (the fI-vE analogy). These analogies, which are shown in Table 2, have various traditional names. For instance, *fE-vI* has names including the Maxwell analogy, direct analogy, impedance analogy, and effort-flow analogy, whereas fI-vE is termed the Firestone analogy, inverse analogy, mobility analogy, and across-through analogy.

One argument in favor of *fE-vI* is that the units of potential [joule per coulombs (volts)] and the units of current [coulombs per second (amperes)] are closely aligned, respectively, with the units of force [joules per meter (newtons)] and the units of velocity (meters per second). Hence, *fE-vI* merely entails replacing meters with coulombs, and transforming mechanical systems to electrical systems involves replacing m, c, and k with L, R, and $1/C$, respectively. Under this analogy, the reference terminal for power transmission matrices is defined in terms of a current level.

In *fI-vE*, force units (newtons) are analogous to current units [coulombs per second (amperes)], and velocity units (meters per second) are analogous to potential units [joule/coulombs (volts)]. The reference terminal for power transmission matrices is defined in terms of potential E. Transforming mechanical systems to electrical systems necessitates replacing m, c, and k with C, $1/R$, and $1/L$, respectively.

There are two reasons why *fE-vI* may seem to be more natural than *fI-vE* as an analogy to mechanical systems. As noted, replacing meters with coulombs transforms force to potential and velocity to current. The analogy between mass and inductance reflects the fact that a mass stores kinetic energy due to velocity v while an inductor stores "kinetic" energy due to the flow of charge (current). Similarly, the analogy between stiffness and the reciprocal of capacitance reflects the storage of potential energy. These energy analogies are less clear for *fI-vE*. On the other hand, the reference terminal for *fE-vI* is a current level, in contrast with the reference terminal for *fI-vE* (which is a potential level). The latter is the natural choice in electrical systems. This point is discussed in [29].

A longstanding difficulty in applying *fI-vE* concerns the ability to realize capacitors with masses. A capacitor that has one terminal grounded can be realized by a mass. The mechanical analogy of an ungrounded capacitor is less obvious. This point is mentioned in [22, p. 20], which states, "Nongrounded capacitors have no mechanical analog." One solution to this problem is discussed in [24, p. 234], which uses a transformer to produce an "ungrounded" mass. The present article takes advantage of the inerter.

Another distinction between *fE-vI* and *fI-vE* is the type of external source that arises in an analogy to external force. In *fE-vI*, the external source is potential, whereas in *fI-vE*, it is current. Converting a structure to a circuit by means of *fE-vI* yields a circuit with an applied potential, while the conversion through *fI-vE* produces a circuit with an applied current.

Two additional mechanical/electrical analogies can be defined for power transmission matrices. Replacing E and $I = \dot{Q}$ in *fE-vI* with Q and $\dot{E}$, respectively, yields fQ-$v\dot{E}$. The reference terminal for power transmission matrices is defined in terms of a potential rate. Although stiffness k is analogous to capacitance, neither mass nor viscosity has analogs in terms of R, L, or C.

TABLE 2 The mechanical-electrical analogies. In analogy *fE-vI*, the external source is potential, and the position and velocity reference terminals are charge and current levels, respectively. In analogy $fQ - v\dot{E}$, the external source is current, and the position and velocity reference terminals are potential and potential-rate levels, respectively. In analogy *fI-vE*, the position and velocity reference terminals are magnetic-flux and potential levels, respectively. In analogy $f\Phi - v\dot{I}$, the position and velocity reference terminals are current and current-rate levels, respectively. Note that, in analogies $fQ - v\dot{E}$ and $f\Phi - v\dot{I}$, m and c have no analogs in terms of R, L, and C. A capacitor may represent a mass or an inerter, depending on whether or not it is shunted to ground.

Mechanical Variables	f	q	v	a	m, b	c	k
Electrical analogy *fE-vI*	E	$Q = \int I$	$\dot{Q} = I$	$\ddot{Q} = \dot{I}$	L	R	$1/C$
Electrical analogy fQ-$v\dot{E}$	$Q = \int I$	E	$\dot{E}$	$\ddot{E}$			C
Electrical analogy *fI-vE*	I	$\Phi = \int E$	$\dot{\Phi} = E$	$\ddot{\Phi} = \dot{E}$	C	$1/R$	$1/L$
Electrical analogy $f\Phi$-$v\dot{I}$	$\Phi = \int E$	I	$\dot{I}$	$\ddot{I}$			L

Alternatively, replacing I and E in fI-vE by $\Phi = \int E$ and $\dot{I}$ in fI-vE, respectively, yields $f\Phi$-$v\dot{I}$, where Φ is magnetic flux with units of webers and 1 Wb = 1 V-s. The reference terminal for power transmission matrices is defined in terms of a current rate. While stiffness k is analogous to inductance, neither mass nor viscosity have analogs in terms of R, L, or C.

It is interesting to note that none of the analogies in Table 2 explicitly requires a reference terminal that is analogous to the unforced particle specified by Newton's second law for force and inertia [21]. In fI-vE, only potential difference is relevant, and there is no need to define an inertial ground. Mass is analogous to capacitance, and the idealized model of a capacitance assumes the ability to send and receive electrons to and from ground without affecting the ground's potential. The electrical ground in a circuit is implicitly assumed to have infinite capacity. Specifically, the earth ground provides an approximately infinite volume that can absorb or lose electrons without losing its charge neutrality. Analogously, in structural applications, an inertially nonrotating massive body provides a reference point for defining the inertial acceleration of a particle subject to forcing. As in the case of the earth ground, whose neutrality is unaffected by accumulating or shedding charge, the inertial velocity of an inertially nonrotating massive body is unaffected by forces. Consequently, under fI-vE, Newton's second law $f = m\mathbf{p}v$ is precisely $I = C\mathbf{p}E$, where an electrical ground with infinite capacity plays the role of an inertial frame.

Although the choice has pros and cons, we adopt the analogy fI-vE. Under it, electrical impedance is the transfer function from current to potential. The analogous transfer function is from force to velocity, which, according to Table 1, is mechanical admittance. Under fI-vE, electrical impedance and admittance are inconsistent with mechanical impedance and admittance. Within the circuit setting, a fourth element can be considered, the memristor, whose mechanical analog is the memdashpot. For details, see "What Is a Memristor?"

POWER TRANSMISSION MATRICES FOR CIRCUITS

Consider the circuit with physical terminals T_1 and T_2 in Figure 16. Under fI-vE, power transmission matrices transform current and potential inputs to current and potential outputs, that is,

What Is a Memristor?

In the analogy fI-vE, force f and velocity v are analogous to current I and potential E, respectively. The integral of force (momentum h) and the integral of velocity (position q) are analogous to charge Q and magnetic flux Φ, respectively. These four mechanical quantities and their electrical analogs can be used to label the four vertices of two squares displayed in Figure S2. In the electrical case, three of the edges correspond to resistors, capacitors, and inductors. The remaining edge can be viewed as the relationship between charge and magnetic flux. It might be expected that the integration operators would cancel and that the remaining edge would be equivalent to resistance. The idealized element of the remaining edge turns out to be a hysteretic operator; this is the memristor [49]. In the mechanical instance, three of the edges correspond to masses, springs, and dashpots. The remaining edge is a hysteretic dashpot called a memdashpot [50], [51], which is the mechanical analog of the memristor.

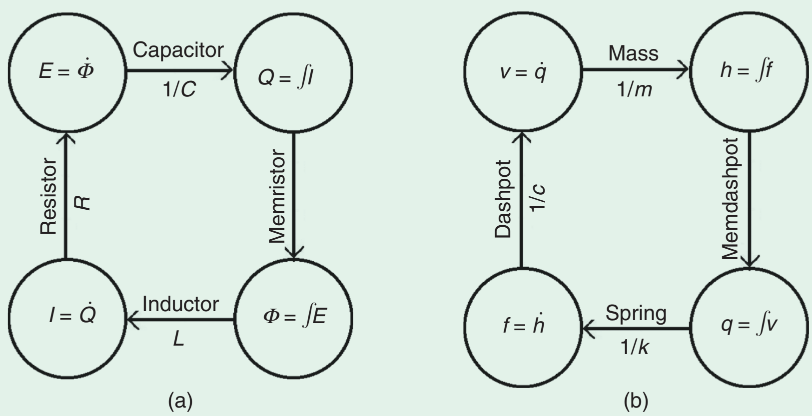

FIGURE S2 A memristor and memdashpot. (a) The four vertices of the square correspond to the four electrical variables that are analogous to force and velocity and their integrals in fI-vE. The four edges of the square correspond to resistors, capacitors, and inductors; the remaining edge is the memristor, which is an idealized hysteretic device. (b) The mechanical analog of the memristor; that is, the memdashpot.

$$\begin{bmatrix} I_2 \\ E_2 \end{bmatrix} = P \begin{bmatrix} I_1 \\ E_1 \end{bmatrix}. \tag{95}$$

When two circuits are joined, the common physical terminals constitute a node through which current passes. The arrows for I_1 and I_2 indicate the direction of current flow in the case where E_1 and E_2 are constant, $E_2 > E_1$, and $I_2 = I_1$; in this instance, $I_1 = I_2 > 0$. In the case where E_1 and E_2 are constant, $E_2 < E_1$, and $I_2 = I_1$, then $I_1 = I_2 < 0$, and the current flows opposite to the indicated direction. This sign convention, which is consistent with the one for forces and structures, avoids the need for the minus signs that appear in the definition of ABCD matrices (as explained in "Power Transmission Matrices and ABCD Parameters").

As a special case, consider the capacitor with physical terminals T_1 and T_2 and reference terminal T_0 that appears in Figure 17(a). For a capacitor with capacitance C, it follows that

$$E_2(t) - E_1(t) = \frac{1}{C\mathbf{p}} I_1(t) = \frac{1}{C\mathbf{p}} I_2(t). \tag{96}$$

Thus the elementary power transmission matrix is given by

$$P_{\mathrm{cap}}(C, \mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{1}{C\mathbf{p}} & 1 \end{bmatrix}. \tag{97}$$

For the resistor with resistance R shown in Figure 17(b), it follows that

$$E_2(t) - E_1(t) = RI_1(t) = RI_2(t) \tag{98}$$

and thus

$$E_2(t) = RI_1(t) + E_1(t). \tag{99}$$

Hence, the elementary power transmission matrix is found by

$$P_{\mathrm{res}}(R, \mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ R & 1 \end{bmatrix}. \tag{100}$$

For the inductor with inductance L shown in Figure 17(c), it follows that

$$E_2(t) - E_1(t) = L\mathbf{p}I_1(t) = L\mathbf{p}I_2(t). \tag{101}$$

Hence, the elementary power transmission matrix is given by

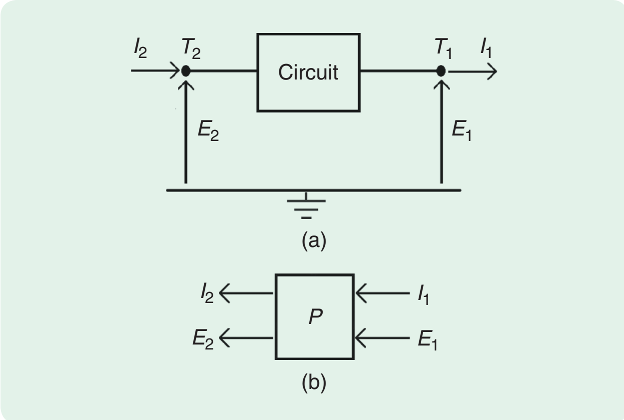

FIGURE 16 (a) A circuit relating current I_1 and potential E_1 to current I_2 and potential E_2. The currents I_1 and I_2, which are absolute (through) variables, flow through the physical terminals T_1 and T_2, respectively, which are nodes. The arrows for I_1 and I_2 indicate the direction of current flow when $E_2 > E_1$ and $I_2 = I_1$; in this case, $I_1 = I_2 > 0$. When $E_2 < E_1$ and $I_2 = I_1$, $I_1 = I_2 < 0$, and the current flows opposite to the indicated direction. The potentials E_1 and E_2 of T_1 and T_2, respectively, are relative (across) variables defined in relation to the reference terminal T_0 (ground). (b) The power transmission matrix P of the circuit relates the current and potential at T_1 to the current and potential at T_2.

Power Transmission Matrices and ABCD Parameters

A related representation of power transmission matrices is given by ABCD parameters, which have the form

$$\begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = P_{\mathrm{ABCD}} \begin{bmatrix} E_1 \\ -I_1 \end{bmatrix}, \tag{S29}$$

where

$$P_{\mathrm{ABCD}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{S30}$$

Consequently, P can be written in terms of ABCD parameters as

$$P = \begin{bmatrix} C & -D \\ A & -B \end{bmatrix}. \tag{S31}$$

The minus sign preceding I_1 reflects the convention that the current is positive in the case where it is flowing into either physical terminal [14, p. 11.6].

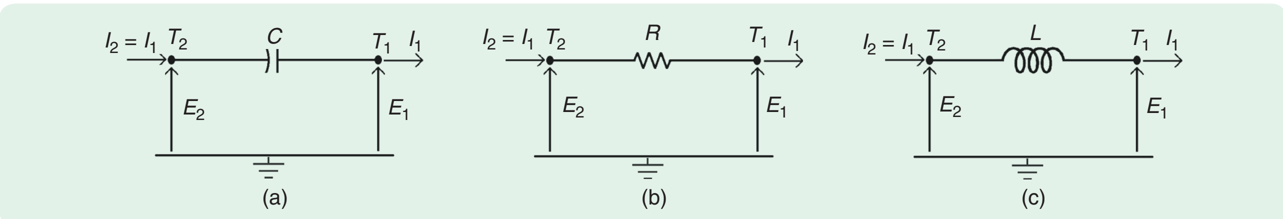

FIGURE 17 (a) A capacitor with capacitance C and current $I_1 = I_2$ flowing through it with potentials E_1 and E_2 at its physical terminals T_1 and T_2, respectively. (b) A resistor with resistance R and current $I_1 = I_2$ flowing through it with potentials E_1 and E_2 at its physical terminals T_1 and T_2, respectively. (c) An inductor with inductance L and current $I_1 = I_2$ flowing through it with potentials E_1 and E_2 at its physical terminals T_1 and T_2, respectively.

$$P_{\mathrm{ind}}(L,\mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ L\mathbf{p} & 1 \end{bmatrix}. \quad (102)$$

Note that power transmission matrices (97), (100), and (102) for the capacitor, resistor, and inductor, respectively, are analogous to power transmission matrices (14), (23), and (19) for the inerter, dashpot, and spring, respectively. To obtain an analog for the mass, we consider the shunted capacitor in Figure 18(a). Letting I_3 denote the current through the capacitor, it follows that $I_2 = I_1 + I_3$. Since $E_1 = E_2$ is the potential drop across the capacitor,

$$I_2 = I_1 + C\mathbf{p}E_1. \quad (103)$$

The elementary power transmission matrix for the shunted capacitor is produced by

$$P_{\mathrm{cap,sh}}(C,\mathbf{p}) \triangleq \begin{bmatrix} 1 & C\mathbf{p} \\ 0 & 1 \end{bmatrix}. \quad (104)$$

Likewise, the elementary power transmission matrices for the shunted resistor and shunted inductor are given by

$$P_{\mathrm{res,sh}}(R) \triangleq \begin{bmatrix} 1 & \frac{1}{R} \\ 0 & 1 \end{bmatrix}, \quad (105)$$

$$P_{\mathrm{ind,sh}}(L,\mathbf{p}) \triangleq \begin{bmatrix} 1 & \frac{1}{L\mathbf{p}} \\ 0 & 1 \end{bmatrix}. \quad (106)$$

Note that the shunted capacitor is analogous to the mass, and the shunted resistor and shunted inductor are analogous to the shunted dashpot and shunted spring, respectively. Observe that none of the circuits in Figure 17 is connected to ground, and T_0 serves as an arbitrary potential reference level; in effect, T_0 may be a floating ground. However, in Figure 18, all three circuits are connected to ground, and T_0 is assumed to be an earth ground, whose potential is constant.

ENERGY TRANSMISSION MATRICES FOR ELECTRICAL SYSTEMS

For energy transmission matrices, the analogies between mechanical and electrical systems require a variable that corresponds to position, where the electrical analog of position is the integral of the electrical analog of velocity. For *fE-vI*, the electrical analog of position is charge; for $fQ\text{-}v\dot{E}$, it is potential; for *fI-vE*, it is magnetic flux; and for $f\Phi\text{-}v\dot{I}$, it is current.

Under *fI-vE*, power transmission matrices transform current and potential inputs into current and potential outputs, according to (95). Energy transmission matrices transform current and magnetic-flux inputs into current and magnetic-flux outputs, that is,

$$\begin{bmatrix} I_2 \\ \Phi_2 \end{bmatrix} = E(\mathbf{p}) \begin{bmatrix} I_1 \\ \Phi_1 \end{bmatrix}. \quad (107)$$

The elementary energy transmission matrices for the capacitor, resistor, and inductor are given by

$$E_{\mathrm{cap}}(C,\mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{1}{C\mathbf{p}^2} & 1 \end{bmatrix}, \quad (108)$$

$$E_{\mathrm{res}}(R,\mathbf{p}) \triangleq \begin{bmatrix} 1 & 0 \\ \frac{R}{\mathbf{p}} & 1 \end{bmatrix}, \quad (109)$$

$$E_{\mathrm{ind}}(L) \triangleq \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix}. \quad (110)$$

RECIPROCITY AND SYMMETRY FOR CIRCUITS

Consider the circuit modeled by the power transmission matrix in Figure 16. Shorting terminal T_2 to ground yields $E_2 = 0$, and (95) becomes

$$\begin{bmatrix} I_2 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ E_1 \end{bmatrix}. \quad (111)$$

The transfer function G_{12} from I_2 to E_1 is given by

$$G_{12} = \frac{p_{21}}{p_{12}p_{21} - p_{11}p_{22}}. \quad (112)$$

Alternatively, shorting T_1 to ground yields $E_1 = 0$, and (95) becomes

$$\begin{bmatrix} I_2 \\ E_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ 0 \end{bmatrix}. \quad (113)$$

The transfer function G_{21} from $-I_1$ to E_2 is given by

$$G_{21} = -p_{21}. \quad (114)$$

The circuit is *reciprocal* if the transfer functions G_{12} and G_{21} are equal, that is, if and only if $\det P = 1$, as in the case of structures. This property is demonstrated in [14, p. 11.3].

A circuit is *symmetric* if reversal of its physical terminals yields the same power transmission matrix.

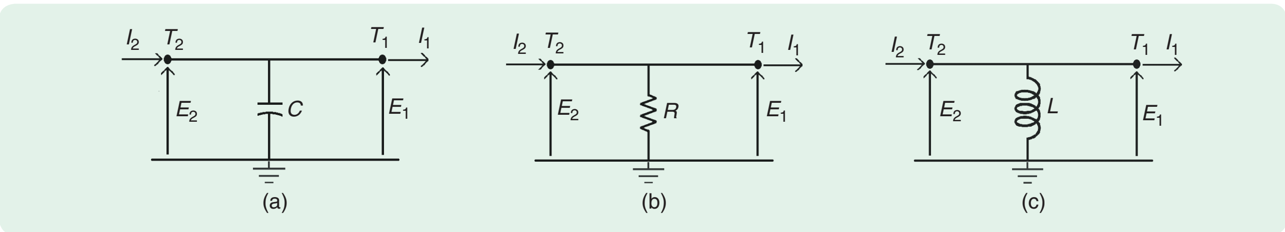

FIGURE 18 (a) A shunted capacitor with capacitance C and currents I_1 and I_2 flowing through its physical terminals T_1 and T_2 with potentials E_1 and E_2, respectively. (b) A shunted resistor with resistance R and currents I_1 and I_2 flowing through its physical terminals T_1 and T_2 with potentials E_1 and E_2, respectively. (c) A shunted inductor with inductance L with currents I_1 and I_2 flowing through its physical terminals T_1 and T_2 with potentials E_1 and E_2, respectively.

As in structures, a circuit is symmetric if and only if the diagonal entries of its power transmission matrix are equal, that is, if and only if the current and potential transmissibilities are equal. Symmetric power transmission matrices play a role in the analysis of transmission lines as described in "Power Transmission Matrices and Transmission Lines."

SERIES AND PARALLEL CONNECTION OF CIRCUITS

The power transmission matrices produced by (97)–(106) are special cases of the circuits shown in Figure 19. The circuit with impedance Z_1 in Figure 19(a) is not grounded. Therefore, $I_2 = I_1$, and the power transmission matrix is given by

$$P_1(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ Z_1(\mathbf{p}) & 1 \end{bmatrix}. \tag{115}$$

Likewise, the power transmission matrix for the shunt circuit with impedance Z_2 in Figure 19(b) is found by

$$P_2(\mathbf{p}) = \begin{bmatrix} 1 & \dfrac{1}{Z_2(\mathbf{p})} \\ 0 & 1 \end{bmatrix}. \tag{116}$$

The circuits in Figure 19(a) and (b) can be connected in series, and the corresponding power transmission matrices are provided by the rules for series connections of structures. Consequently, the power transmission matrix for the circuit in Figure 19(c) is calculated by

$$P_1(\mathbf{p})P_2(\mathbf{p}) = \begin{bmatrix} 1 & \dfrac{1}{Z_2(\mathbf{p})} \\ Z_1(\mathbf{p}) & 1+\dfrac{Z_1(\mathbf{p})}{Z_2(\mathbf{p})} \end{bmatrix}, \tag{117}$$

whereas the power transmission matrix for the circuit in Figure 19(d) results from

$$P_2(\mathbf{p})P_1(\mathbf{p}) = \begin{bmatrix} 1+\dfrac{Z_1(\mathbf{p})}{Z_2(\mathbf{p})} & \dfrac{1}{Z_2(\mathbf{p})} \\ Z_1(\mathbf{p}) & 1 \end{bmatrix}. \tag{118}$$

TRANSFORMING STRUCTURES INTO CIRCUITS AND VICE VERSA

One of the benefits of analogies is the ability to convert a structure into a dynamically equivalent circuit and vice versa. As depicted in [20], this can be done under *fE-vI* by

Power Transmission Matrices and Transmission Lines

One of the main applications of power transmission matrices is to approximate the modeling of transmission lines [5, Ch. 1, pp. 200–218], [12, pp. 212–214], [13, pp. 361, 362]. Although a transmission is a distributed electrical element, an approximate model can be constructed by viewing the transmission line as the series connection of n identical power transmission matrices. Assuming that each circuit in the series connection is symmetric, it follows that

$$\begin{bmatrix} I_n \\ E_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & A \end{bmatrix}^n \begin{bmatrix} I_1 \\ E_1 \end{bmatrix}. \tag{S32}$$

Defining

$$a \triangleq \cosh^{-1} A \tag{S33}$$

$$Z_0 \triangleq \sqrt{B/C}, \tag{S34}$$

it follows that [12, p. 213]

$$\begin{bmatrix} A & B \\ C & A \end{bmatrix} = \begin{bmatrix} \cosh a & Z_0 \sinh a \\ \dfrac{\sinh a}{Z_0} & \cosh a \end{bmatrix} \tag{S35}$$

and thus [52, p. 32]

$$\begin{bmatrix} A & B \\ C & A \end{bmatrix}^n = \begin{bmatrix} \cosh an & Z_0 \sinh an \\ \dfrac{\sinh an}{Z_0} & \cosh an \end{bmatrix}. \tag{S36}$$

If the transmission line is terminated with the impedance Z_0, the input impedance Z_{in} of the transmission line is produced by [12, p. 214]

$$Z_{\text{in}} = \frac{E_n \cosh an + Z_0 I_n \sinh an}{(E_n/Z_0)\sinh an + I_n \cosh an}. \tag{S37}$$

These expressions can be used to analyze wave propagation in transmission lines [5, Ch. 1, pp. 200–218], [12, pp. 214–217].

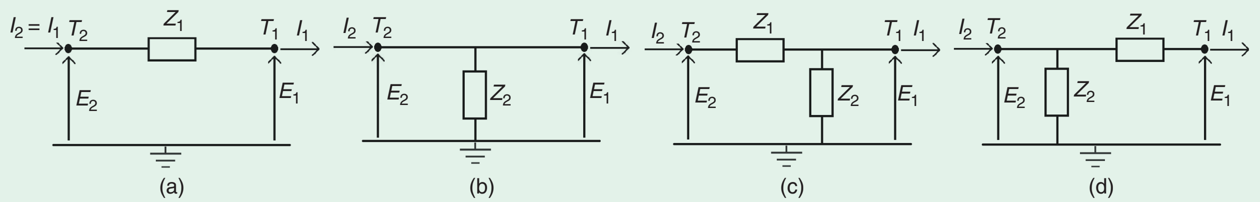

FIGURE 19 (a) An ungrounded circuit with impedance Z_1. (b) A shunted circuit with impedance Z_2. (c) An ungrounded circuit with impedance Z_1 in series with a shunted circuit with impedance Z_2. (d) An ungrounded circuit with impedance Z_1 in series with a shunted circuit with impedance Z_2, where the order of the circuits is reversed relative to (c).

replacing series connections with parallel connections and parallel connections with series connections. Under *fI-vE*, the conversion is more direct, with series connections of structures replaced by series connections of circuits and likewise for parallel connections. To demonstrate that feature, the circuit analogs of Examples 3 and 4 are constructed. In these examples, external forces are replaced by current sources.

Example 6: Circuit Analog of the Single-Degree-of-Freedom Structure

Under *fI-vE*, Figure 20 presents the circuit analog of the single-degree-of-freedom structure from Figure 9(a). The inerter b, spring k, and dashpot c in parallel in Figure 9(a) are replaced by the capacitor C, inductor L, and resistor R, respectively. The RLC components are connected in series with the capacitor C_0, which is shunted to ground in analogy to the mass m. Furthermore, the external force f is replaced by the current source I. Therefore,

$$P(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ \dfrac{\mathbf{p}}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} & 1 \end{bmatrix}. \tag{119}$$

Note that the parallel connection of the capacitor, inductor, and resistor is linked in series with capacitor C_0 (which is shunted to ground), as shown in Figure 19. Using the boundary conditions $E_1 = 0$ and $I_2 = I$ (where I_2 denotes the current flowing through T_2), it follows from (118) that

$$\begin{bmatrix} I(t) \\ E_2(t) \end{bmatrix} = \begin{bmatrix} 1 & C_0\mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{\mathbf{p}}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} & 1 \end{bmatrix} \begin{bmatrix} I_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} I_1(t) + \dfrac{C_0\mathbf{p}^2}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} I_1(t) \\ \dfrac{\mathbf{p}}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} I_1(t) \end{bmatrix}, \tag{120}$$

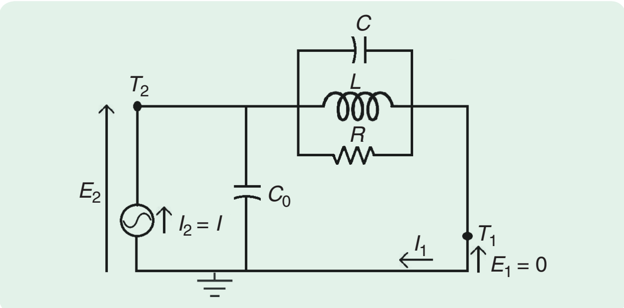

FIGURE 20 A circuit analog of the single-degree-of-freedom structure shown in Figure 9. The capacitors C and C_0, the inductor L, and the resistor R in parallel are analogous to the inerter, mass, spring, and dashpot in parallel. The current source plays the role of the external force.

where I_1 denotes the current flowing through T_1. Next,

$$I(t) = \frac{(C_0 + C)\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} I_1(t), \tag{121}$$

$$E_2(t) = \frac{\mathbf{p}}{C\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} I_1(t). \tag{122}$$

Combining (121) and (122) yields

$$E_2(t) = \frac{\mathbf{p}}{(C_0 + C)\mathbf{p}^2 + \dfrac{\mathbf{p}}{R} + \dfrac{1}{L}} I(t). \tag{123}$$

Example 7: Circuit Analog of a Single-Degree-of-Freedom Structure

Consider the single-degree-of-freedom structure detailed in Figure 21(a), where the mass and inerter are in series, with a spring and dashpot in parallel. The transmission matrix of the parallel spring–dashpot connection results from

$$P(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ \dfrac{\mathbf{p}}{c\mathbf{p} + k} & 1 \end{bmatrix}. \tag{124}$$

Moreover, the parallel spring–dashpot connection is linked in series with the inerter and the mass, and thus,

$$\begin{bmatrix} f(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \dfrac{mb\mathbf{p}^2 + c(m+b)\mathbf{p} + k(m+b)}{b(c\mathbf{p} + k)} & m\mathbf{p} \\ \dfrac{b\mathbf{p}^2 + c\mathbf{p} + k}{b\mathbf{p}(c\mathbf{p} + k)} & 1 \end{bmatrix} \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}. \tag{125}$$

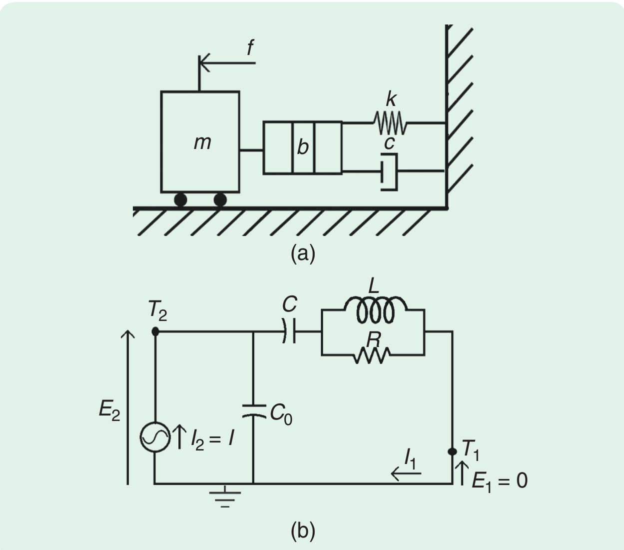

FIGURE 21 (a) A single-degree-of freedom structure with a mass and inerter in series with a spring and dashpot in parallel. (b) The analogous circuit.

It follows from (125) that

$$f(t) = \frac{mb\mathbf{p}^2 + c(m+b)\mathbf{p} + k(m+b)}{b(c\mathbf{p} + k)} f_1(t), \tag{126}$$

$$v(t) = \frac{b\mathbf{p}^2 + c\mathbf{p} + k}{b\mathbf{p}(c\mathbf{p}+k)} f_1(t), \tag{127}$$

and thus

$$v(t) = \frac{b\mathbf{p}^2 + c\mathbf{p} + k}{\mathbf{p}(bm\mathbf{p}^2 + (b+m)c\mathbf{p} + k(b+m))} f(t). \tag{128}$$

Under *fI-vE*, Figure 21(b) shows the circuit analog of the single-degree-of-freedom structure displayed in Figure 21(a). The parallel connection of the spring k and dashpot c, which are connected in series with the inerter b in Figure 21(a), is replaced by the parallel connection of the inductor L and resistor R, which are connected in series with the capacitor C. The RLC components are connected in parallel with capacitor C_0, which is shunted to ground in analogy to the mass m. The external force f is replaced by the current source I. In analogy to (65), where b, k, and c are analogous to C, $1/L$, and $1/R$, respectively, the power transmission matrix P of the capacitor-inductor-resistor connection results from

$$P(\mathbf{p}) = \begin{bmatrix} 1 & 0 \\ \dfrac{CLR\mathbf{p}^2 + L\mathbf{p} + R}{C\mathbf{p}(R+L\mathbf{p})} & 1 \end{bmatrix}. \tag{129}$$

Note that the capacitor-inductor-resistor connection is in series to capacitor C_0 (which is shunted to ground), as in Figure 19. Using the boundary conditions $E_1 = 0$ and $I_2 = I$, it follows from (118) that

$$\begin{bmatrix} I(t) \\ E_2(t) \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{C_0\mathbf{p}(CLR\mathbf{p}^2 + L\mathbf{p} + R)}{C\mathbf{p}(R+L\mathbf{p})} & C_0\mathbf{p} \\ \dfrac{CLR\mathbf{p}^2 + L\mathbf{p} + R}{C\mathbf{p}(R+L\mathbf{p})} & 1 \end{bmatrix} \begin{bmatrix} I_1(t) \\ 0 \end{bmatrix}. \tag{130}$$

It thus follows that

$$I(t) = \frac{CC_0LR\mathbf{p}^2 + (C+C_0)L\mathbf{p} + (C+C_0)R}{C(R+L\mathbf{p})} I_1(t), \tag{131}$$

$$E_2(t) = \frac{CLR\mathbf{p}^2 + L\mathbf{p} + R}{C\mathbf{p}(R+L\mathbf{p})} I_1(t). \tag{132}$$

Substituting I_1 from (132) into (131) produces

$$E_2(t) = \frac{C\mathbf{p}^2 + \frac{1}{R}\mathbf{p} + \frac{1}{L}}{\mathbf{p}(CC_0\mathbf{p}^2 + (C+C_0)\frac{1}{R}\mathbf{p} + (C+C_0)\frac{1}{L})} I(t). \tag{133}$$

Example 8: Circuit Analog of a Two-Degrees-of-Freedom Structure

Under *fI-vE*, Figure 22 displays the circuit analog of the of the two-degrees-of-freedom structure in Figure 11(a). Masses m_1 and m_2 and springs k_1 and k_2 are replaced by shunted capacitors C_1 and C_2 and inductors L_1 and L_2, respectively, which are connected in series. The external force f is replaced by the current source I, which is added between C_1 and L_2. In analogy to (72), where m_1, m_2, k_1, and k_2 correspond to C_1, C_2, $1/L_1$, and $1/L_2$, respectively, we have (134), shown at the bottom of the page, and thus,

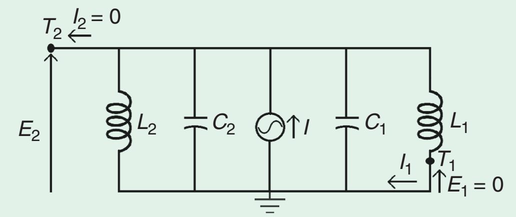

FIGURE 22 The circuit analog of the two-degrees-of-freedom structure in Figure 11. The capacitors C_1 and C_2 and inductors L_1 and L_2 in parallel are analogous to the masses m_1 and m_2 and the springs k_1 and k_2, respectively. The current source I applied between capacitor C_1 and inductor L_2 is analogous to the external force f applied between mass m_1 and spring k_2.

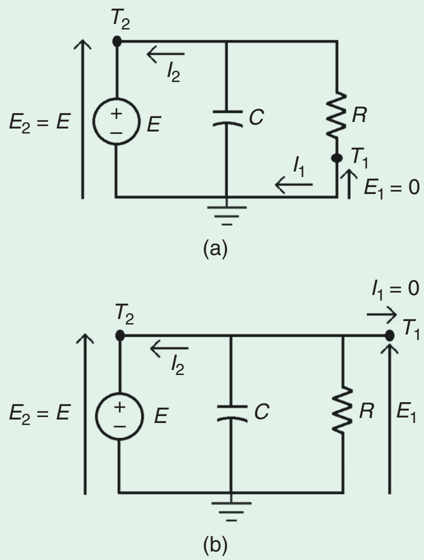

FIGURE 23 A resistor-capacitator circuit with a voltage source. In (a), only the capacitor is modeled as shunted; in (b), both the resistor and capacitor are modeled as shunted.

$$\begin{aligned} \begin{bmatrix} 0 \\ E_2(t) \end{bmatrix} &= \begin{bmatrix} C_2L_2\mathbf{p}^2+1 & C_2\mathbf{p} \\ L_2\mathbf{p} & 1 \end{bmatrix} \left(\begin{bmatrix} C_1L_1\mathbf{p}^2+1 & C_1\mathbf{p} \\ L_1\mathbf{p} & 1 \end{bmatrix} \begin{bmatrix} I_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} -I(t) \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} -(C_2L_2\mathbf{p}^2+1)I(t) + (C_1C_2L_1L_2\mathbf{p}^4 + C_1L_1\mathbf{p}^2 + C_2L_2\mathbf{p}^2 + C_2L_1\mathbf{p}^2+1)I_1(t) \\ -L_2\mathbf{p}I + \mathbf{p}(C_1L_1L_2\mathbf{p}^2 + L_1 + L_2)I_1(t) \end{bmatrix}, \end{aligned} \tag{134}$$

$$0 = -(C_2L_2\mathbf{p}^2+1)I(t) + (C_1C_2L_1L_2\mathbf{p}^4 + C_1L_1\mathbf{p}^2 + C_2L_2\mathbf{p}^2 + C_2L_1\mathbf{p}^2 + 1)I_1(t) \quad (135)$$

$$E_2(t) = -L_2(t)\mathbf{p}I + \mathbf{p}(C_1L_1L_2\mathbf{p}^2 + L_1 + L_2)I_1(t). \quad (136)$$

Solving (135) for I_1 yields

$$I_1(t) = \frac{C_2L_2\mathbf{p}^2+1}{C_1C_2L_1L_2\mathbf{p}^4 + C_1L_1\mathbf{p}^2 + C_2L_1\mathbf{p}^2 + C_2L_2\mathbf{p}^2 + 1} I(t). \quad (137)$$

Substituting (137) in (136) produces

$$E_2(t) = \frac{\frac{1}{L_2}\mathbf{p}}{\left(C_1\mathbf{p}^2 + \frac{1}{L_1}\right)\left(C_2\mathbf{p}^2 + \frac{1}{L_2}\right) + \frac{C_2}{L_2}\mathbf{p}^2} I(t), \quad (138)$$

which is analogous to (76). ◇

Example 9: Circuit With a Voltage Source

Consider the resistor-capacitator circuit in Figure 23, where the goal is to determine the transfer function from the input voltage E to current I_2. Only the capacitor is modeled as shunted with the boundary conditions $E_1 = 0$ and $E_2 = E$, as seen in Figure 23(a). Using the corresponding power transmission matrices yields

$$\begin{bmatrix} I_2 \\ E \end{bmatrix} = \begin{bmatrix} 1 & C\mathbf{p} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ R & 1 \end{bmatrix}\begin{bmatrix} I_1 \\ 0 \end{bmatrix}. \quad (139)$$

Therefore,

$$I_2(t) = (1 + RC\mathbf{p})I_1(t), \quad (140)$$

$$E(t) = RI_1(t). \quad (141)$$

Substituting I_1 from (141) into (140) yields

$$I_2(t) = \left(\frac{1}{R} + C\mathbf{p}\right)E(t). \quad (142)$$

Alternatively, the capacitor and resistor are both modeled as shunted with the boundary conditions $E_2 = E$ and $I_1 = 0$, shown in Figure 23(b). Using the corresponding power transmission matrices produces

$$\begin{bmatrix} I_2 \\ E \end{bmatrix} = \begin{bmatrix} 1 & C\mathbf{p} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{R} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ E_1 \end{bmatrix}. \quad (143)$$

Therefore,

$$I_2(t) = \left(\frac{1}{R} + C\mathbf{p}\right)E_1(t), \quad (144)$$

$$E(t) = E_1(t). \quad (145)$$

Substituting E_1 from (145) into (144) yields (142). ◇

CONCLUSIONS

Transmission matrices relate pairs of power-conjugate variables, where one variable is absolute and the other variable is relative. This article presents a self-contained treatment of the application of power and energy transmission matrices to modeling mechanical systems composed of masses, inerters, springs, and dashpots. The examples illustrate the role of feedback connections in deriving transfer functions. The across-through analogy between structures and circuits is used to obtain analogous results for electrical systems. Through further analogies, this technique can be applied to acoustic and thermal systems.

We believe that power and energy transmission matrices provide an elegant approach to modeling structures and circuits. Although this technique is at least 80 years old, it has gained very little traction in absolute terms and relative to the literature on bond graphs. One reason for this deficit is that the development of transmission matrices in structures and circuits has largely occurred independently, and a complete treatment is not available. Another possible reason, as we have found, is that care is needed in working with transmission matrices to understand the roles of inertial and electrical grounds, include the inerter, and formulate conventions for obtaining the correct signs that respect Newton's third law and the direction of current flow.

It is interesting that circuits are composed of three elements (resistors, inductors, and capacitors), whereas structures are composed of four elements (masses, inerters, springs, and dashpots), with the restriction that a mass cannot be placed in parallel with an inerter, spring, or dashpot. This discrepancy can be traced to the fact that ball-screw and rack-and-pinion inerters take advantage of coupling between translational and rotational motion, which has no counterpart in electrical components. Under the analogy *fI-vE*, a capacitor may represent a mass or an inerter; likewise, under the analogy *fE-vI*, an inductor may represent those components. The analogies between structures and circuits possess subtleties and surprises that warrant future investigation.

In view of the analogy between structures and circuits (not to mention acoustics and thermodynamics), it is reasonable to expect that modeling techniques applicable in one domain are applicable in other ones. For example, Kirchhoff's current law has the mechanical interpretation that the summation of forces acting on a particle with zero mass is zero. Under the analogy *fI-vE*, Newton's second law is equivalent to the capacitor equation $I = C\dot{E}$. Along the same lines, Lagrangian dynamics can be used to model circuits [48]. These and other ramifications of dynamical analogies remain to be fully explored to facilitate control-system applications.

ACKNOWLEDGMENTS

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REFERENCES

[1] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Englewood Cliffs, NJ: Prentice Hall, 1973.

[2] F. T. Brown, *Engineering System Dynamics: A Unified Graph-Centered Approach*, 2nd ed. New York: Taylor & Francis, 1998.

[3] R. Collier, *Transmission Lines: Equivalent Circuits, Electromagnetic Theory, and Photons*. Cambridge, U.K.: Cambridge Univ. Press, 2013.

[4] C. D. Ferris, *Linear Network Theory*. Columbus, OH: C. E. Merrill Books, 1962.

[5] M. S. Ghausi and J. J. Kelly, *Introduction to Distributed Parameter Networks with Application to Integrated Circuits*. New York: Holt, Rinehart, and Winston, 1968.

[6] W. W. Harman and D. W. Lytle, *Electrical and Mechanical Networks: An Introduction to Their Analysis*. New York: McGraw-Hill, 1962.

[7] C. M. Harris, *Shock and Vibration Handbook*, 3rd ed. New York: McGraw-Hill, 1988.

[8] H. E. Koenig and W. A. Blackwell, *Electromechanical System Theory*. New York: McGraw-Hill, 1961.

[9] V. H. Neubert, *Mechanical Impedance: Modelling/Analysis of Structures*. State College, PA: Jostens, 1987.

[10] R. W. Newcomb, *Linear Multiport Synthesis*. New York: McGraw-Hill, 1966.

[11] E. C. Pestel and F. A. Leckie, *Matrix Methods in Elastomechanics*. New York: McGraw-Hill, 1963.

[12] L. A. Pipes and L. R. Harvill, *Applied Mathematics for Engineers and Physicists*, 3rd ed. New York: Dover, 2014.

[13] L. A. Pipes, *Matrix Methods for Engineering*. Englewood Cliffs, NJ: Prentice Hall, 1963.

[14] R. R. Singh, *Electrical Networks*. New York: McGraw-Hill, 2009.

[15] M. E. Van Valkenburg, *Network Analysis*, 3rd ed. Englewood Cliffs, NJ: Prentice Hall, 1974.

[16] P. E. Wellstead, *Introduction to Physical System Modelling*. New York: Academic, 1979.

[17] Anonymous, "Two-Port Networks," 2019. [Online]. Available: https://en.wikipedia.org/wiki/Two-port_network

[18] H. F. Olson, *Dynamic Analogies*. New York: Van Nostrand, 1943.

[19] W. J. Karplus, W. W. Soroka, *Analog Methods: Computation and Simulation*, 2nd ed. New York: McGraw-Hill, 1959.

[20] P. Gardonio and M. J. Brennan, "On the origins and development of mobility and impedance methods in structural dynamics," *J. Sound Vib.*, vol. 249, no. 3, pp. 557–573, 2002.

[21] D. S. Bernstein, "Newton's frames," *IEEE Contr. Syst. Mag.*, vol. 28, pp. 17–18, Feb. 2008.

[22] I. J. Busch-Vishniac, *Electromechanical Sensors and Actuators*. New York: Springer-Verlag, 1999.

[23] P. P. L. Regtien, *Sensors for Mechatronics*, 3rd ed. Amsterdam, The Netherlands: Elsevier, 2012.

[24] J. L. Shearer, A. T. Murphy, and, H. H. Richardson, *Introduction to System Dynamics*. Reading, MA: Addison-Wesley, 1967.

[25] C. W. De Silva, *Mechatronics: An Integrated Approach*. Boca Raton, FL: CRC, 2005.

[26] E. Skudrzyk, *Simple and Complex Vibratory Systems*. Philadelphia, PA: Univ. of Pennsylvania Press, 1968.

[27] J. C. Willems, "The behavioral approach to open and interconnected systems," *IEEE Contr. Syst. Mag.*, vol. 27, no. 6, pp. 46–99, 2007.

[28] J. C. Willems, "Terminals and ports," *IEEE Circ. Syst. Mag.*, vol. 10, no. 4, pp. 8–26, 2010.

[29] J. Miles, "Applications and limitations of mechanical-electrical analogies, new and old," *J. Acoust. Soc. Amer.*, vol. 14, no. 3, pp. 183–192, 1943.

[30] S. Rubin, "Mechanical immittance- and transmission-matrix concepts," *J. Acoust. Soc. Amer.*, vol. 41, no. 5, pp. 1171–1179, 1967.

[31] G. J. O'Hara, "Mechanical impedance and mobility concepts," *J. Acoust. Soc. Amer.*, vol. 41, no. 5, pp. 1180–1184, 1967.

[32] F. E. Cellier, *Continuous System Modeling*. New York: Springer-Verlag, 1991.

[33] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, eds., *Modeling and Control of Complex Physical Systems: The Port Hamiltonian Approach*. New York: Springer-Verlag, 2009.

[34] P. J. Gawthrop and G. P. Bevan, "Bond-graph modeling," *IEEE Contr. Syst. Mag.*, vol. 27, no. 6, pp. 24–45, 2007.

[35] L. Ljung and T. Glad, *Modeling of Dynamic Systems*. Englewood Cliffs, NJ: Prentice Hall, 1994.

[36] M. C. Smith, "Synthesis of mechanical networks: The inerter," *IEEE Trans. Autom. Contr.*, vol. 47, no. 10, pp. 1648–1662, 2002.

[37] M. Z. Q. Chen, C. Papageorgiou, F. Scheibe, F.-C. Wang, and M. C. Smith, "The missing mechanical circuit element," *IEEE Circ. Syst. Mag.*, vol. 9, no. 1, pp. 10–26, 2009.

[38] M. Z. Q. Chen and Y. Hu, *Inerter and Its Application in Vibration Control Systems*. New York: Springer-Verlag, 2019.

[39] K. F. Aljanaideh and D. S. Bernstein, "Initial conditions in time- and frequency-domain system identification: Implications of the shift operator versus the Z transform," *IEEE Contr. Syst. Mag.*, vol. 38, no. 2, pp. 80–93, 2018.

[40] N. Maia, J. Silva, and A. Ribeiro, "The transmissibility concept in multi-degree-of-freedom systems," *Mech. Syst. Sig. Proc.*, vol. 15, no. 1, pp. 129–137, 2001.

[41] C. Rainieri and G. Fabbrocino, *Operational Modal Analysis of Civil Engineering Structures: An Introduction and Guide for Applications*. New York: Springer-Verlag, 2014.

[42] C. W. De Silva, ed., *Vibration and Shock Handbook*. Boca Raton, FL: CRC, 2005.

[43] K. F. Aljanaideh and D. S. Bernstein, "A behavioral equation framework for time-domain transmissibilities," *Automatica*, vol. 78, pp. 20–24, Apr. 2017.

[44] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*. New York: Springer-Verlag, 1998.

[45] J. Y. Ha and K. Kim, "Analysis of MIMO mechanical systems using the vectorial 4 pole parameter method," *J. Sound Vib.*, vol. 180, no. 2, pp. 333–350, 1995.

[46] J. C. Snowdon, "Mechanical four-pole parameters: Transmission matrices," Pennsylvania State Univ. Park Appl. Res. Lab, Tech. Rep., 1976.

[47] L. A. Pipes, "The matrix theory of four-terminal networks," *Philos. Mag. Series 7*, vol. 30, no. 202, pp. 370–395, 1940.

[48] D. Jeltsema and J. M. A. Scherpen, "Multidomain modeling of nonlinear networks and systems," *IEEE Contr. Syst. Mag.*, vol. 29, pp. 28–59, Aug. 2009.

[49] L. O. Chua, "Memristor-the missing circuit element," *IEEE Trans. Circ. Theory*, vol. 18, no. 5, pp. 507–519, 1971.

[50] A. Franze, "Extending the classification of devices in single-degree-of-freedom vibrating systems," *Proc. Appl. Math. Mech.*, vol. 16, no. 1, pp. 269–270, 2016.

[51] M. E. Fouda, A. G. Radwan, A. S. Elwakil, and N. K. Nawayseh, "Review of the missing mechanical element: Memdamper," in *Proc. IEEE Int. Conf. Electronics, Circuits, and Systems*, 2015, pp. 201–204.

[52] V. Pop and O. Furdui, *Square Matrices of Order 2*. New York: Springer-Verlag, 2017.