Demystifying Enigmatic Undershoot in Setpoint Command Following

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One of the most striking phenomena in systems and control theory is *initial undershoot*, where the initial direction of the response of a system to a step input or setpoint command is opposite to the direction of the asymptotic response. As mentioned in "Summary," initial undershoot represents a fundamental limitation on the performance of a control system, and it has potentially serious consequences, especially when the initially "wrong" direction violates constraints on the output of the system. For example, in motion control applications, such as robotics or autonomous vehicles, initial undershoot may result in a collision. In any event, initial undershoot is an intriguing, visual example of a *system-theoretic phenomenon* that occurs in electrical, mechanical, and cyberphysical systems, and it is one of the many deleterious effects of nonminimum-phase zeros on the achievable performance of feedback control systems [1]–[3].

The study of initial undershoot has an interesting and somewhat convoluted history. According to [4], initial undershoot is discussed in [5] (in Japanese) for single-input, single-output (SISO) continuous-time systems, where it is shown that initial undershoot occurs if and only if the system has an odd number of real zeros that are greater than zero. This result is later reported by many researchers for continuous-time systems, including [6]–[11]. In [12] and [13], initial undershoot is investigated for transfer functions that arise in process control. The results of [5] are extended to multiple-input, multiple-output continuous-time systems in [4], where it is shown that initial undershoot is not directly related to the transmission zeros of the transfer function matrix.

Initial undershoot also occurs in discrete-time systems. This case is discussed in [14] and [15], where it is stated without proof that initial undershoot occurs if and only if the discrete-time system has an odd number of real zeros that are greater than one.

Although initial undershoot occurs in the discretized dynamics of sampled-data systems, the situation is more complicated due to the fact that the zeros of a sampled-data system depend on the poles and zeros of the underlying continuous-time system as well as the sample time [16], [17, p. 63]. In addition, a discretized plant with sufficiently small sample time may possess sampling zeros, although these are negative [17, p. 64], [18]. The relationship between the zeros of the discretized plant and the zeros of the continuous-time plant is, therefore, significantly more complicated than the exponential map, which relates the poles of the discretized plant to the poles of the continuous-time system independently of the zeros. Despite this complication, the step response of a sampled-data system is simply a sampled version of the step response of the continuous-time system. Therefore, under sufficiently fast sampling, the discretized system has initial undershoot if and only if the continuous-time system has initial undershoot, and, thus, the number of real zeros greater than one in the discretized system is odd if and only if the number of real zeros greater than zero in the continuous-time system is odd.

Summary

It often occurs in practice that the response to a step input or setpoint command moves initially in a direction that is opposite to the direction of the asymptotic response. In many real-world applications, this phenomenon—called initial undershoot—presents a fundamental limitation on control system performance. Although the basic mechanism responsible for initial undershoot, namely, an odd number of real, positive zeros, is well understood, it is surprising that, as the setpoint changes, initial undershoot may appear or disappear for the same plant dynamics. The goal of this tutorial note is to investigate the causes of this puzzling phenomenon. In particular, for setpoint command following with a changing setpoint, this article shows (spoiler alert) that the internal state when the setpoint changes determines the presence or absence of initial undershoot. Complete proofs for both initial and delayed undershoot in both continuous time and discrete-time systems are given to make the article self-contained and useful for students and instructors of systems and control theory.

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Under sufficiently fast sampling, the zeros of a discretized system are approximately equal to $e^{z_i T_s}$, where z_i are the zeros of the continuous-time system. Moreover, under sufficiently fast sampling, all of the sampling zeros are negative. Therefore, under sufficiently fast sampling, the number of positive zeros of the continuous-time system is equal to the number of real zeros greater than one in the discretized system. This observation is given in Theorem 9. On the other hand, under sufficiently slow sampling, the discretization of a strictly proper system with nonzero dc gain does not have initial undershoot and, thus, as stated in Theorem 10, has no real zeros greater than one.

An extension of initial undershoot is the phenomenon of *delayed undershoot* (also called *inverse response*), where, at some time after the onset of a step input or setpoint command, the direction of the system response relative to the initial response is opposite to the direction of the asymptotic response relative to the initial response. Initial undershoot can thus be viewed as an extreme case of delayed undershoot, where the "delay" is zero. Since oscillatory systems typically exhibit delayed undershoot, however, this phenomenon cannot be tied solely to the zeros of the system [19]. Nevertheless, the existence of at least one real zero greater than zero is a sufficient condition for delayed undershoot, which is consistent with the fact that, if the step response has initial undershoot, then it also has delayed undershoot.

Having established the fundamental mechanisms underlying initial and delayed undershoot, this article focuses on the puzzling situation where, during the operation of a closed-loop system with a changing setpoint command, the response may exhibit initial undershoot in certain instances but not others. Surprisingly, this can occur despite the fact that the plant is unchanged, and there are no external disturbances. The main contribution of this article is to investigate the underlying cause of this "enigmatic" undershoot. In particular, this article shows that initial and delayed undershoot are not determined solely by the transfer function of the forced response of the system but, rather, depend on a modified transfer function that depends on the initial state of a realization of the plant. Examples 4 and 12 in this article show that a nonzero initial state may induce initial undershoot in a plant whose forced response alone does not have initial undershoot, and vice versa. Within a control system context, where a setpoint command plays the role of a step input, it turns out that, if the internal state converges after a change in the setpoint, then the initial and delayed undershoot behaviors are consistent for each new setpoint. If, however, the setpoint command changes before the internal state converges, then the initial or delayed undershoot behavior may change depending on the internal state when the setpoint command changes. Since, in practice, the internal state is unknown, initial or delayed undershoot may occur without explanation. The main contribution of this article is, thus, to demystify enigmatic undershoot.

PRELIMINARIES FOR CONTINUOUS-TIME SYSTEMS

Consider the continuous-time system

$$\dot{x}_c(t) = A_c x_c(t) + B_c \bar{u}, \tag{1}$$

$$y_c(t) = C_c x_c(t) + E_c \bar{u}, \tag{2}$$

where for all $t \geq 0$, $x_c(t) \in \mathbb{R}^n$ is the state, $y_c(t) \in \mathbb{R}$ is the output, and $\bar{u} \in \mathbb{R}$ is a nonzero step input. The matrices (A_c, B_c, C_c, E_c) are assumed to be a minimal state-space realization of the transfer function

$$G_c(s) \triangleq C_c(sI - A_c)^{-1} B_c + E_c. \tag{3}$$

Note that $G_c(\infty) = E_c$.

Throughout this article, $G_c(s)$ is assumed to be asymptotically stable, SISO, and of the form

$$G_c(s) = K\frac{N_c(s)}{D_c(s)} = K\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}, \tag{4}$$

where $n \geq 1$, $n \geq m \geq 0$, $K \neq 0$, and $z_1, \ldots, z_m \in \mathbb{C}$ and $p_1, \ldots, p_n \in \mathbb{C}$ are the zeros and poles of $G_c(s)$, respectively. Since (A_c, B_c, C_c) is controllable and observable, the polynomials N_c and D_c have no common roots. Let $d \triangleq n - m \geq 0$ denote the relative degree of $G_c(s)$. Note that $G_c(s)$ is strictly proper, that is, $d > 0$, if and only if $E_c = 0$, and $G_c(s)$ is exactly proper, that is, $d = 0$, if and only if $E_c = K$. If $G_c(s)$ is exactly proper, then $\hat{G}_c(s) \triangleq G_c(s) - E_c$ is the *strictly proper part* of $G_c(s)$, which is given by

$$\hat{G}_c(s) = K\frac{N_c(s)}{D_c(s)} - K = K\frac{N_c(s) - D_c(s)}{D_c(s)}. \tag{5}$$

Since N_c and D_c have no common roots, it follows that $N_c - D_c$ and D_c have no common roots, and, thus, there is no pole-zero cancellation in (5). Furthermore, $N_c - D_c$ and N_c have no common roots, and, thus, the zeros of $\hat{G}_c(s)$ are different from the zeros of $G_c(s)$. If $G_c(s)$ is strictly proper, then $\hat{G}_c(s) = G_c(s)$.

For all $t \geq 0$, the step response of (1) and (2) with a possibly nonzero initial state $x_c(0)$ is

$$y_c(t) = C_c e^{tA_c} x_c(0) + \int_0^t C_c e^{(t-\tau)A_c} B_c \bar{u}\, \mathrm{d}\tau + E_c \bar{u}. \tag{6}$$

Hence,

$$y_c(0) = C_c x_c(0) + E_c \bar{u}. \tag{7}$$

If $x_c(0) = 0$ and $E_c = 0$, then $y_c(0) = 0$.

Definition 1

The response $y_c(t)$ given by (6) has *initial undershoot* if there exists $t_1 > 0$ such that, for all $t \in (0, t_1)$,

$$[y_c(t) - y_c(0)][y_c(\infty) - y_c(0)] \leq 0. \tag{8}$$

Definition 2

The response $y_c(t)$ given by (6) has *delayed undershoot* if there exists $t > 0$ such that (8) holds.

The distinction between initial undershoot and delayed undershoot (Table 1) is explained in "Initial Undershoot Versus Delayed Undershoot." Note that if $y_c(t)$ has initial undershoot, then $y_c(t)$ has delayed undershoot. However, Example 6 shows that the converse is not true. Moreover, if $y_c(\infty) = y_c(0)$, then y_c has initial and delayed undershoot. The special case $y_c(0) = 0$ is worth noting.

Proposition 1

Assume that $y_c(0) = 0$. Then, $y_c(t)$ has initial undershoot if and only if there exists $t_1 > 0$ such that, for all $t \in (0, t_1)$,

$$y_c(t)\, y_c(\infty) \le 0. \tag{9}$$

In addition, $y_c(t)$ has delayed undershoot if and only if there exists $t > 0$ such that (9) holds.

The following observation is immediate but worth noting.

Proposition 2

Let $\alpha \in \mathbb{R}$. Then, $y_c(t)$ has initial undershoot if and only if $y_c(t) - \alpha$ has initial undershoot. In addition, $y_c(t)$ has delayed undershoot if and only if $y_c(t) - \alpha$ has delayed undershoot.

TABLE 1 A summary of continuous-time definitions and results.

Definition 1	Initial undershoot
Definition 2	Delayed undershoot
Proposition 1	Initial undershoot with the zero initial state
Proposition 2	Shift-invariance property of initial and delayed undershoot
Proposition 3	Application of Proposition 2 for exactly proper systems
Proposition 4	Laplace transform of $y_c(t) - y_c(0)$
Proposition 5	Asymptotic value $y_c(\infty)$ of $y_c(t)$
Proposition 6	First nonzero right-sided derivative of $y_c(t)$ at $t = 0$
Proposition 7	Initial sign of $y_c(t)$
Proposition 8	Initial undershoot with the zero initial state using relative degree
Theorem 1	Initial undershoot with the zero initial state using zeros
Theorem 2	Initial undershoot with a nonzero initial state
Theorem 3	Delayed undershoot
Theorem 4	Lower bound on the maximum deviation of $y_c(t)$ from $y_c(0)$

Proof

For all $t \ge 0$, define $y_{c,\alpha}(t) \triangleq y_c(t) - \alpha$ and note that

$$[y_{c,\alpha}(t) - y_{c,\alpha}(0)][y_{c,\alpha}(\infty) - y_{c,\alpha}(0)] = [y_c(t) - y_c(0)][y_c(\infty) - y_c(0)]. \tag{10}$$

Hence, (8) is satisfied if and only if

$$[y_{c,\alpha}(t) - y_{c,\alpha}(0)]\,[y_{c,\alpha}(\infty) - y_{c,\alpha}(0)] \le 0. \qquad \square$$

Proposition 2 shows that initial undershoot is preserved under an arbitrary, constant offset of the step response. The following result views $E_c \bar{u}$ as a constant offset of the step response of the strictly proper part of $G_c(s)$.

Proposition 3

Let $x_c(0) \in \mathbb{R}^n$. Then, $y_c(t)$ has initial undershoot if and only if $y_c(t) - E_c \bar{u}$ has initial undershoot. In addition, $y_c(t)$ has

Initial Undershoot Versus Delayed Undershoot

The following example illustrates the distinction between initial and delayed undershoot.

Example S1

Consider the continuous-time transfer functions

$$G_c(s) = \frac{-(s-1)}{(s+1)^3}, \tag{S1}$$

$$G_c(s) = \frac{(s-1)^2}{(s+1)^3}. \tag{S2}$$

Assuming that the free response is zero, the unit step responses $y_c(t)$ of (S1) and (S2) have, respectively, initial and delayed undershoot, as shown in Figure S1. $\diamond$

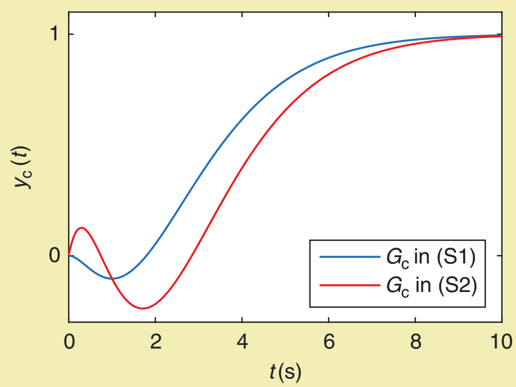

FIGURE S1 In Example S1, the unit step responses $y_c(t)$ of (S1) and (S2) have, respectively, initial and delayed undershoot.

delayed undershoot if and only if $y_c(t)-E_c\bar{u}$ has delayed undershoot.

In view of (6), Proposition 3 shows that, for all initial states $x_c(0)$, the presence or absence of initial undershoot is independent of the value of E_c. Hence, there is no loss of generality by setting $E_c=0$ in (2), that is, by replacing the exactly proper transfer function $G_c(s)$ with its strictly proper part $\hat{G}_c(s)$. This observation is further justified by the following result, which shows that the Laplace transform of $y_c(t)-y_c(0)$ does not depend on E_c.

Proposition 4

Let $y_c(t)$ be the step response of (1) and (2) given by (6). Then,

$$\mathcal{L}\{y_c(t)-y_c(0)\}=\tilde{G}_c(s)\frac{\bar{u}}{s}, \tag{11}$$

where

$$\tilde{G}_c(s)\triangleq C_c(sI-A_c)^{-1}\Big[\frac{1}{\bar{u}}A_c x_c(0)+B_c\Big]. \tag{12}$$

Proof

Subtracting (7) from (6) and taking the Laplace transform yields

$$\begin{aligned}\mathcal{L}\{y_c(t)-y_c(0)\}&=C_c(sI-A_c)^{-1}x_c(0)+C_c(sI-A_c)^{-1}B_c\frac{\bar{u}}{s}\\&\quad-\frac{1}{s}C_c x_c(0)\\&=C_c(sI-A_c)^{-1}\Big[x_c(0)+B_c\frac{\bar{u}}{s}\Big]-\frac{1}{s}C_c x_c(0)\\&=C_c(sI-A_c)^{-1}\Big[x_c(0)+B_c\frac{\bar{u}}{s}-\frac{1}{s}(sI-A_c)x_c(0)\Big]\\&=C_c(sI-A_c)^{-1}\Big[\frac{1}{s}B_c\bar{u}+\frac{1}{s}A_c x_c(0)\Big]\\&=\tilde{G}_c(s)\frac{\bar{u}}{s}.\end{aligned}$$

□

The situation is different, however, for the case of the nonzero initial state since, as shown by (6), the effect of $x_c(0)$ is not equivalent to a constant offset of $y_c(t)$. Examples 4 and 6 show that initial undershoot or delayed undershoot may occur for some initial states but not others. The following result, which follows from the final value theorem [20, p. 15], [21], provides an expression for the asymptotic value $y_c(\infty)$ of $y_c(t)$.

Proposition 5

$y_c(\infty)\triangleq\lim_{t\to\infty}y_c(t)$ exists and is given by

$$y_c(\infty)=\lim_{s\to 0}G_c(s)\bar{u}=G_c(0)\bar{u}=K\frac{N_c(0)}{D_c(0)}\bar{u}. \tag{13}$$

In view of (8), Proposition 5 implies that if

$$G_c(0)\bar{u}=C_c x_c(0)+E_c\bar{u} \tag{14}$$

then $y_c(t)$ has initial undershoot. In particular, in the special case where $x_c(0)=0$, Proposition 5 implies that if

$$G_c(0)=E_c, \tag{15}$$

then $y_c(t)$ has initial undershoot. However, Example 4 shows that the converses of these statements are not true.

INITIAL UNDERSHOOT FOR CONTINUOUS-TIME SYSTEMS WITH ZERO INITIAL STATE

For the case of zero initial state, this section provides a necessary and sufficient condition for initial undershoot in terms of the zeros of the strictly proper part $\hat{G}_c(s)$ of $G_c(s)$. The following section considers the case of the nonzero initial state.

The right-sided derivative of $y_c(t)$ at $t=0$ is given by

$$y_c'(0^+)\triangleq\lim_{t\downarrow 0}\frac{y_c(t)-y_c(0)}{t}. \tag{16}$$

Since y_c' is continuous, it follows that

$$y_c'(0^+)=\lim_{t\downarrow 0}y_c'(t)=\lim_{t\downarrow 0}\lim_{\delta\to 0}\frac{y_c(t+\delta)-y_c(t)}{\delta}. \tag{17}$$

Likewise, for all $i\in\{1,2,3,\ldots\}$, define the ith right-sided derivative of $y_c(t)$ at $t=0$ by

$$y_c^{(i)}(0^+)\triangleq\lim_{t\downarrow 0}\frac{\mathrm{d}^i y_c(t)}{\mathrm{d}t^i}. \tag{18}$$

The following result, which follows from the initial value theorem [22, p. 816], concerns the first nonzero right-sided derivative of $y_c(t)$ at $t=0$.

Proposition 6

Assume that $G_c(s)$ is strictly proper and $x_c(0)=0$. Then, for all $i\in\{1,\ldots,d-1\}$, $y_c^{(i)}(0^+)=\lim_{s\to\infty}s^i G_c(s)\bar{u}=0$. Furthermore, $y_c^{(d)}(0^+)=\lim_{s\to\infty}s^d G_c(s)\bar{u}=K\bar{u}$.

Proposition 7

Assume that $G_c(s)$ is strictly proper and $x_c(0)=0$. Then, there exists $t_1>0$ such that, for all $t\in(0,t_1)$,

$$\operatorname{sign} y_c(t)=\operatorname{sign} y_c^{(d)}(0^+)=\operatorname{sign} K\bar{u}. \tag{19}$$

Proof

Note that, for all $t>0$, $y_c(t)$ is a finite sum $\sigma(t)$ of sinusoidal and exponential functions; this sum is the real-analytic extension of $y_c(t)$ to $\mathbb{R}$. Using Proposition 6, it follows that for all $t\in\mathbb{R}$,

$$\begin{aligned}\sigma(t)&=\sigma(0)+\sum_{i=1}^{\infty}\frac{\sigma^{(i)}(0)t^i}{i!}\\&=y_c(0)+\sum_{i=1}^{d}\frac{y_c^{(i)}(0^+)t^i}{i!}+\sum_{i=d+1}^{\infty}\frac{\sigma^{(i)}(0)t^i}{i!}\\&=\frac{K\bar{u}t^d}{d!}+\sum_{i=d+1}^{\infty}\frac{\sigma^{(i)}(0)t^i}{i!},\end{aligned}$$

where, as $t\to 0$,

$$\sum_{i=d+1}^{\infty}\frac{\sigma^{(i)}(0)t^i}{i!}=O(t^{d+1}).$$

Therefore, as $t \to 0$,

$$\sigma(t) = \frac{K\bar{u}t^d}{d!} + O(t^{d+1}),$$

Thus, as $t \downarrow 0$,

$$y_c(t) = \frac{K\bar{u}t^d}{d!} + O(t^{d+1}).$$

Hence, there exists $t_1 > 0$ such that, for all $t \in (0, t_1)$, (19) is satisfied. □

The following result follows from Propositions 1 and 7.

Proposition 8

Assume that $x_c(0) = 0$, and let $\hat{d} \geq 1$ denote the relative degree of $\hat{G}_c(s)$. Then, the step response $y_c(t)$ of $G_c(s)$ has initial undershoot if and only if

$$y_c^{(\hat{d})}(0^+)\,[y_c(\infty) - E_c\bar{u}] \leq 0. \tag{20}$$

Proof

Since $y_c(t) - E_c\bar{u}$ is the step response of $\hat{G}_c(s)$, Propositions 1 and 7 imply that $y_c(t) - E_c\bar{u}$ has initial undershoot if and only if (20) is satisfied. Thus, Proposition 2 implies that $y_c(t)$ has initial undershoot if and only if (20) is satisfied. □

Theorem 1

Assume that $x_c(0) = 0$ and $G_c(0) \neq E_c$. Then, the step response $y_c(t)$ of $G_c(s)$ has initial undershoot if and only if $\hat{G}_c(s)$ has an odd number of real zeros greater than zero.

Proof

First, consider the case where $E_c = 0$; that is, $G_c(s)$ is strictly proper. Thus, Propositions 5 and 6 imply that

$$y_c^{(d)}(0^+)\,y_c(\infty) = K^2\bar{u}^2\frac{N_c(0)}{D_c(0)}. \tag{21}$$

Since $G_c(0) \neq 0$, it follows that $N_c(0) \neq 0$. Moreover, since $G_c(s)$ is asymptotically stable, every real root of D_c is negative. Therefore, using Lemma S1 (see "Determining the Sign of a Polynomial"), it follows from (21) that

$$\begin{aligned}\operatorname{sign}[y_c^{(d)}(0^+)\,y_c(\infty)] &= \operatorname{sign}\left(K^2\bar{u}^2\frac{N_c(0)}{D_c(0)}\right)\\ &= \operatorname{sign}\frac{(-1)^{\pi_{N_c}(0)}}{(-1)^{\pi_{D_c}(0)}}\\ &= \operatorname{sign}\frac{(-1)^{\pi_{N_c}(0)}}{(-1)^0} = (-1)^{\pi_{N_c}(0)},\end{aligned}$$

where $\pi_p(\alpha)$ is the number, counting multiplicity, of real roots greater than $\alpha \in \mathbb{R}$ of the polynomial p. Thus, $y_c^{(d)}(0^+)\,y_c(\infty) < 0$ if and only if $\pi_{N_c}(0)$ is odd. Therefore, Proposition 8 implies that $y_c(t)$ has initial undershoot if and only if $\pi_{N_c}(0)$ is odd, that is, if and only if $G_c(s) = \hat{G}_c(s)$ has an odd number of real zeros greater than zero.

Next, consider the case where $E_c \neq 0$. Since $y_c(t) - E_c\bar{u}$ is the step response of the strictly proper transfer function $\hat{G}_c(s)$, it follows that $y_c(t) - E_c\bar{u}$ has initial undershoot if and only if $\hat{G}_c(s)$, has an odd number of real zeros greater than zero. Thus, Proposition 3 implies that $y_c(t)$ has initial undershoot if and only if $\hat{G}_c(s)$ has an odd number of real zeros greater than zero. □

The following example (Table 2) illustrates Propositions 5–8 and Theorem 1.

Example 1

Let $x_c(0) = 0$ and $\bar{u} = 1$, and consider the transfer function

$$G_c(s) = \frac{-200\,(s-1)}{(s+1)\,(s+2)\,(s+3)\,(s+4)}, \tag{22}$$

which has exactly one real zero greater than zero. Hence, Theorem 1 implies that $y_c(t)$ has initial undershoot.

To show this more directly, Proposition 6 implies that

$$y_c^{(1)}(0^+) = \lim_{s\to\infty} sG_c(s) = 0, \tag{23}$$

$$y_c^{(2)}(0^+) = \lim_{s\to\infty} s^2G_c(s) = 0, \tag{24}$$

$$y_c^{(3)}(0^+) = \lim_{s\to\infty} s^3G_c(s) = -200. \tag{25}$$

Alternatively,

$$\begin{aligned} y_c(t) &= \mathcal{L}^{-1}\left\{\frac{G_c(s)}{s}\right\}\\ &= 100\mathcal{L}^{-1}\left\{\frac{1}{12s} - \frac{2}{3(s+1)} + \frac{3}{2(s+2)} - \frac{4}{3(s+3)}\right.\\ &\quad \left. + \frac{5}{12(s+4)}\right\}\\ &= 100\left(\frac{1}{12} - \frac{2}{3}e^{-t} + \frac{3}{2}e^{-2t} - \frac{4}{3}e^{-3t} + \frac{5}{12}e^{-4t}\right), \end{aligned} \tag{26}$$

and thus,

$$y_c^{(1)}(t) = 100\left(\frac{2}{3}e^{-t} - 3e^{-2t} + 4e^{-3t} - \frac{5}{3}e^{-4t}\right), \tag{27}$$

$$y_c^{(2)}(t) = -\,100\left(\frac{2}{3}e^{-t} + 6e^{-2t} - 12e^{-3t} + \frac{20}{3}e^{-4t}\right), \tag{28}$$

$$y_c^{(3)}(t) = 100\left(\frac{2}{3}e^{-t} - 12e^{-2t} + 36e^{-3t} - \frac{80}{3}e^{-4t}\right). \tag{29}$$

TABLE 2 A summary of continuous-time examples.

Example S1	Distinction between initial and delayed undershoot
Example 1	Initial undershoot with the zero initial state
Example 2	Initial undershoot for a two-link planar system
Example 3	Zeros of exactly proper systems
Example 4	Initial undershoot with a nonzero initial state
Example 5	Delayed undershoot
Example 6	Delayed undershoot
Example 7	Application of Theorem 4
Example 8	Initial undershoot in setpoint command following
Example 9	Enigmatic undershoot in setpoint command following

It follows from (27)–(29) that $y_c^{(1)}(0^+) = y_c^{(2)}(0^+) = 0$ and $y_c^{(3)}(0^+) = -200$, as shown in (23)–(25). Figure 1 shows the unit step response $y_c(t)$ of (22). Note that, as implied by Proposition 7, for all sufficiently small $t > 0$, $\operatorname{sign} y_c(t) = \operatorname{sign} y_c^{(d)}(0^+)$. Moreover, Proposition 5 yields $y_c(\infty) = \lim_{t\to\infty} y_c(t) = \lim_{s\to 0} G_c(s) = G_c(0) = 25/3$, as shown in Figure 1. Since $y_c^{(d)}(0^+) y_c(\infty) < 0$, it follows from Proposition 8 that $y_c(t)$ has initial undershoot. ◊

Example 2

Consider the planar two-link system shown in Figure 2, where p_1 is the point where the first link is connected to the fixed horizontal plane, p_2 is the point where the first link is connected to the second link, $\ell_1 = \ell_2 = 0.5$ m are the length of the links, $m_1 = m_2 = 10$ kg are the masses of the links, $c_1 = 10$ kg·m²/rad and $c_2 = 2$ kg·m²/rad are the dampings at the joints, $k_1 = 5$ N·m/rad and $k_2 = 1$ N·m/rad are

Determining the Sign of a Polynomial

Lemma S1

Let $p(x) = \beta_n x^n + \cdots + \beta_1 x + \beta_0$ be a nonzero polynomial with real coefficients; assume that $\beta_n \neq 0$; let $\alpha \in \mathbb{R}$; assume that $p(\alpha) \neq 0$; and let $\pi_p(\alpha)$ be the number, counting multiplicity, of real roots of p greater than α. Then,

$$\operatorname{sign} p(\alpha) = (-1)^{\pi_p(\alpha)} \operatorname{sign} \beta_n. \quad \text{(S3)}$$

Proof

Let $\sigma_1, \ldots, \sigma_{\pi_p(\alpha)}$ denote the real roots of p that are greater than α; let $\rho_1, \ldots, \rho_\ell$ denote the real roots of p that are less than α; and let $\xi_1, \bar{\xi}_1, \ldots, \xi_r, \bar{\xi}_r$ denote the nonreal complex roots of p, where $\bar{\xi}$ denotes the complex conjugate of $\xi \in \mathbb{C}$. Note that $n = \pi_p(\alpha) + \ell + 2r$. Define

$$p_1(x) \triangleq (x - \sigma_1) \cdots (x - \sigma_{\pi_p(\alpha)}),$$
$$p_2(x) \triangleq (x - \rho_1) \cdots (x - \rho_\ell),$$
$$p_3(x) \triangleq (x - \xi_1)(x - \bar{\xi}_1) \cdots (x - \xi_r)(x - \bar{\xi}_r),$$

and note that

$$\operatorname{sign} p_1(\alpha) = \operatorname{sign}(\alpha - \sigma_1) \cdots \operatorname{sign}(\alpha - \sigma_{\pi_p(\alpha)}) = (-1)^{\pi_p(\alpha)}, \quad \text{(S4)}$$

$$\operatorname{sign} p_2(\alpha) = \operatorname{sign}(\alpha - \rho_1) \cdots \operatorname{sign}(\alpha - \rho_\ell) = 1, \quad \text{(S5)}$$

$$\operatorname{sign} p_3(\alpha) = \operatorname{sign}[(\alpha - a_1)^2 + b_1^2] \cdots \operatorname{sign}[(\alpha - a_r)^2 + b_r^2] = 1, \quad \text{(S6)}$$

where, for all $i = 1, \ldots, r$, $a_i \triangleq \operatorname{Re} \xi_i$ and $b_i \triangleq \operatorname{Im} \xi_i$. Thus, since $p(\alpha) = p_1(\alpha) p_2(\alpha) p_3(\alpha) \beta_n$, it follows from (S4)–(S6) that $\operatorname{sign} p(\alpha) = (\operatorname{sign} p_1(\alpha))(\operatorname{sign} p_2(\alpha))(\operatorname{sign} p_3(\alpha)) \operatorname{sign} \beta_n = (-1)^{\pi_p(\alpha)} \operatorname{sign} \beta_n$. □

Replacing x by $-x$ in Lemma S1 yields the following result.

Lemma S2

Let $p(x) = \beta_n x^n + \cdots + \beta_1 x + \beta_0$, where $\beta_n \neq 0$, be a nonzero polynomial with real coefficients; let $\alpha \in \mathbb{R}$; assume that $p(\alpha) \neq 0$; and let $\nu_p(\alpha)$ be the number, counting multiplicity, of real roots of p less than α. Then,

$$\operatorname{sign} p(\alpha) = (-1)^{n + \nu_p(\alpha)} \operatorname{sign} \beta_n. \quad \text{(S7)}$$

Note that (S3) and (S7) imply that

$$\operatorname{par} \pi_p(\alpha) = \operatorname{sign} \frac{p(\alpha)}{\beta_n}, \quad \text{(S8)}$$

$$\operatorname{par} \nu_p(\alpha) = \operatorname{sign} \frac{(-1)^n p(\alpha)}{\beta_n}, \quad \text{(S9)}$$

where "par" denotes parity. Note that (S8) and (S9) provide expressions for the parities of $\pi_p(\alpha)$ and $\nu_p(\alpha)$ that do not require knowledge of the roots of p.

Example S2

Let $p(x) \triangleq -2x^5 + 24x^4 - 90x^3 + 120x^2 - 88x + 96$, and note that $p(0) = 96$, $p(3) = -60$, $p(5) = 156$, and $p(7) = -1500$. Thus, it follows from (S8) that

$$\operatorname{par} \pi_p(0) = -1, \quad \operatorname{par} \pi_p(3) = 1, \quad \text{(S10)}$$

$$\operatorname{par} \pi_p(5) = -1, \quad \operatorname{par} \pi_p(7) = 1. \quad \text{(S11)}$$

In fact, since $\operatorname{roots}(p) = \{2, 4, 6, \pm \jmath\}$, it follows that

$$\pi_p(0) = 3, \quad \pi_p(3) = 2, \quad \text{(S12)}$$

$$\pi_p(5) = 1, \quad \pi_p(7) = 0, \quad \text{(S13)}$$

which implies (S11). Figure S2 shows $p(\alpha)$ versus α. ◊

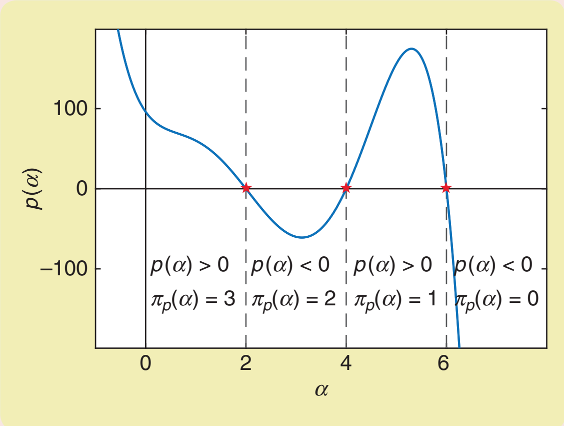

FIGURE S2 Example S2. $p(\alpha)$ versus α. The red stars show the real roots of p.

the stiffnesses of the joints, and θ_1 and θ_2 are the angles from the fixed horizontal plane to the links. Let u_1 and u_2 denote the external torques applied to joints p_1 and p_2, respectively. The linearized equation of motion about the equilibrium $[\theta_1\ \ \theta_2\ \ \dot\theta_1\ \ \dot\theta_2]^T = 0$ is given by

$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \ddot\theta_1\\ \ddot\theta_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -5.14 & 1.71 & -10.29 & 3.43\\ 8.91 & -3.77 & 17.83 & -7.54\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \dot\theta_1\\ \dot\theta_2\end{bmatrix} + \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0.69 & -1.03\\ -1.03 & 2.74\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}. \tag{30}$$

See [23] for more details. The transfer function from u_1 to θ_2 is given by

$$G_{\theta_2 u_1}(s) = \frac{-1.03\,(s+0.4)\,(s-2)}{(s+16.34)\,(s+0.52)\,(s^2+0.98s+0.49)}, \tag{31}$$

which has exactly one real zero greater than zero. Note that [23] shows that, for a two-link system, the linearized transfer function from u_1 to θ_2 has exactly one real zero greater than zero, regardless of the values of the parameters. Theorem 1 thus implies that the unit step response $y_c(t)$ of (31) with the zero initial state has initial undershoot, as shown in Figure 3. ◊

In the case where $x_c(0) = 0$, Theorem 1 provides necessary and sufficient conditions for the existence of initial undershoot. Although $G_c(s)$ may be either strictly proper or exactly proper, this necessary and sufficient condition concerns the strictly proper part $\hat{G}_c(s)$ of $G_c(s)$. In the case where $G_c(s)$ is exactly proper, $G_c(s)$ has n zeros, and $\hat{G}_c(s)$ has $m < n$ zeros. This leads to the question as to whether or not the presence or absence of initial undershoot can be directly characterized in terms of the zeros of $G_c(s)$ rather than indirectly in terms of the zeros of $\hat{G}_c(s)$. The following example investigates this question.

Example 3

Let E_c be a nonzero real number, and let $G_c(s)$ be the transfer function

$$G_c(s) = \frac{s-1}{(s+2)^2} + E_c = \frac{E_c s^2 + (4E_c+1)s + 4E_c - 1}{(s+2)^2}, \tag{32}$$

whose strictly proper part has exactly one real zero greater than zero. Furthermore,

$$G_c(s) = \frac{(s-z_1)(s-z_2)}{(s+2)^2}, \tag{33}$$

where, for $E_c \neq 0$,

$$z_1 = \frac{-4E_c - 1 - \sqrt{12E_c+1}}{2E_c}, \tag{34}$$

$$z_2 = \frac{-4E_c - 1 + \sqrt{12E_c+1}}{2E_c}. \tag{35}$$

Therefore, if $E_c \in (-\infty, -1/12)$, then z_1 and z_2 are complex conjugates, whereas if $E_c = -1/12$, then $z_1 = z_2 = 4$. Furthermore, as E_c increases from $-1/12$ to zero, z_1 and z_2 depart from four in opposite directions, with $\lim_{E_c\uparrow 0} z_1 = 1$ and $\lim_{E_c\uparrow 0} z_2 = \infty$. Finally, as E_c increases from zero to ∞, z_1 decreases from one to -2, and z_2 increases from $-\infty$ to -2. Therefore, although the strictly proper part of $G_c(s)$ has exactly one real zero greater than zero, $G_c(s)$ may have zero real zeros, exactly two real zeros less than zero, exactly one real zero greater than zero, or exactly two real zeros

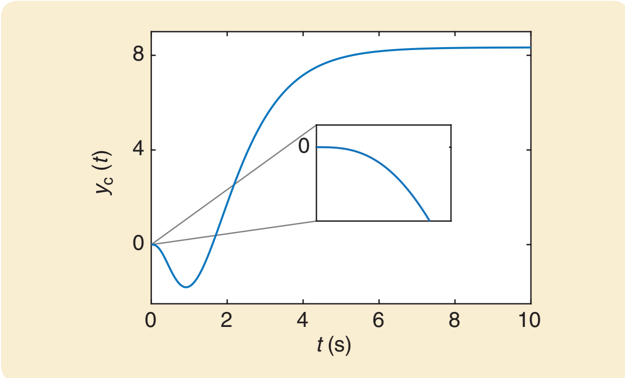

FIGURE 1 Example 1. The unit step response of (22). Note that $y_c^{(1)}(0^+) = 0$ and $y_c^{(2)}(0^+) = 0$. However, $y_c^{(3)}(0^+) < 0$, and thus, for all sufficiently small $t > 0$, $y_c(t) < 0$. Furthermore, since the asymptotic response is $y_c(\infty) = 25/3 > 0$, $y_c(t)$ has initial undershoot. Alternatively, since $G_c(s)$ has exactly one real zero greater than zero, $y_c(t)$ has initial undershoot.

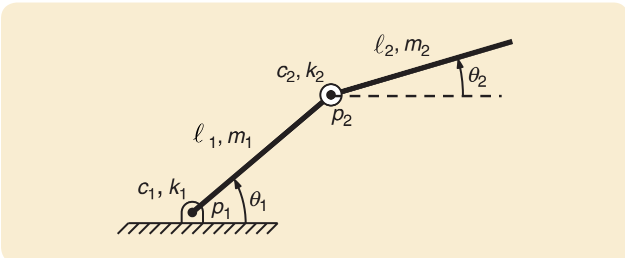

FIGURE 2 Example 2. A two-link planar system.

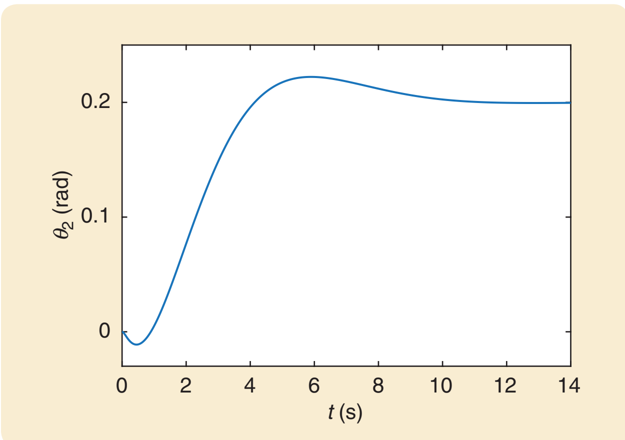

FIGURE 3 Example 2. For the two-link system shown in Figure 2, the transfer function in (31) has exactly one real zero greater than zero, and thus, the unit step response θ_2 of (31) has initial undershoot.

greater than zero. Therefore, the zeros of an exactly proper transfer function cannot be used to infer the presence or absence of initial undershoot. ◊

INITIAL UNDERSHOOT FOR CONTINUOUS-TIME SYSTEMS WITH A NONZERO INITIAL STATE

For the case where the initial state is not necessarily zero, this section provides a necessary and sufficient condition for initial undershoot in terms of the zeros of $\tilde{G}_c(s)$ defined by (12). The following result is a consequence of Theorem 1 and Propositions 2 and 4.

Theorem 2

Assume that $x_c(0) \in \mathbb{R}^n$ and $\tilde{G}_c(0) \neq 0$, where $\tilde{G}_c(s)$ is given by (12). Then, the step response $y_c(t)$ has initial undershoot if and only if $\tilde{G}_c(s)$ has an odd number of real zeros greater than zero.

Proof

For the realization $(A_c, (1/\bar{u})A_c x_c(0) + B_c, C_c)$ of $\tilde{G}_c(s)$, Proposition 4 implies that

$$\dot{\tilde{x}}_c(t) = A_c \tilde{x}_c(t) + \left[\frac{1}{\bar{u}} A_c x_c(0) + B_c\right]\bar{u},$$
$$y_c(t) - y_c(0) = C_c \tilde{x}_c(t),$$

where the internal state $\tilde{x}_c(t) \in \mathbb{R}^n$ satisfies $\tilde{x}_c(0) = 0$. Thus, since $\tilde{G}_c(s)$ is strictly proper, and $\tilde{G}_c(0) \neq 0$, Theorem 1 implies that $y_c(t) - y_c(0)$ has initial undershoot if and only if $\tilde{G}_c(s)$ has an odd number of real zeros greater than zero. Therefore, Proposition 2 implies that $y_c(t)$ has initial undershoot if and only if $\tilde{G}_c(s)$ has an odd number of real zeros greater than zero. □

Theorem 2 corrects [24, Th. III.9], where the term $G_c(s) - y_c(0)$ appears in place of $\tilde{G}_c(s)$. Note that, if $x_c(0) = 0$, then (13) implies that $\tilde{G}_c(s) = \hat{G}_c(s)$; otherwise, $\tilde{G}_c(s)$ and $\hat{G}_c(s)$ may have different zeros, as demonstrated by the following example.

Example 4

Let

$$G_c(s) = \frac{100(s-1)(s-3)(s+6)}{(s+2)^3}, \tag{36}$$

which has a minimal realization $(A_c, B_c, C_c, 100)$, where

$$A_c = \begin{bmatrix} -6 & -3 & -2 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 32 \\ 0 \\ 0 \end{bmatrix}, \tag{37}$$

$$C_c = \frac{1}{32}[-400 \ \ -825 \ \ 250]. \tag{38}$$

Note that

$$\hat{G}_c(s) = G_c(s) - 100 = \frac{-400(s+8.54)(s-0.29)}{(s+2)^3}, \tag{39}$$

and thus, $\hat{d} = 1$. Let $\bar{u} = 1$, and consider the step response $y_c(t)$ with the two initial states $x_c(0) = 0$, and $x_c(0) = [0\ 20\ 0]^T$. Note that, for $x_c(0) = 0$, (12) implies that $\tilde{G}_c(s) = \hat{G}_c(s)$. Thus, for $x_c(0) = 0$, $\tilde{G}_c(s)$ has exactly one real zero greater than zero, whereas for $x_c(0) = [0\ 20\ 0]^T$, (12) implies that

$$\tilde{G}_c(s) = \frac{2025(s+7.12)(s+1.42)}{4(s+2)^3}, \tag{40}$$

which has no real zeros greater than zero. Theorem 2 implies that $y_c(t)$ has initial undershoot with $x_c(0) = 0$, but it does not have initial undershoot with $x_c(0) = [0\ 20\ 0]^T$, as shown in Figure 4. ◊

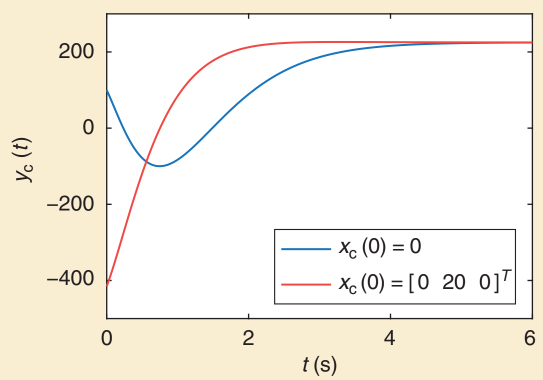

FIGURE 4 Example 4. The response $y_c(t)$ of (36) for two initial states. With $x_c(0) = 0, \tilde{G}_c(s) = \hat{G}_c(s)$ has exactly one real zero greater than zero, and, thus, $y_c(t)$ has initial undershoot, whereas with $x_c(0) = [0\ \ 20\ \ 0]^T, \tilde{G}_c(s)$ has no real zeros greater than zero, and thus, $y_c(t)$ does not have initial undershoot.

DELAYED UNDERSHOOT FOR CONTINUOUS-TIME SYSTEMS

For the case where the initial state is not necessarily zero, this section provides a sufficient condition for delayed undershoot in terms of the zeros of $\tilde{G}_c(s)$ defined by (12).

Theorem 3

Let $x_c(0) \in \mathbb{R}^n$, define $\tilde{G}_c(s)$ by (12), assume that $\tilde{G}_c(0) \neq 0$, and assume that $\tilde{G}_c(s)$ has at least one real zero greater than zero. Then, the response $y_c(t)$ has delayed undershoot.

Proof

Let $z_1 \in (0, \infty)$ be a real zero of $\tilde{G}_c(s)$. Proposition 4 implies that

$$\int_0^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,\mathrm{d}t = \tilde{G}_c(z_1)\frac{\bar{u}}{z_1} = 0, \tag{41}$$

where, since $\tilde{G}_c(s)$ is asymptotically stable, and $z_1 > 0$, the integral is convergent. Since $e^{-z_1 t} > 0$ for all $t \geq 0$, (41) implies that there exist $t_1 > 0$ and $t_2 > 0$ such that $y_c(t_1) - y_c(0) > 0$ and $y_c(t_2) - y_c(0) < 0$. Moreover, since $\tilde{G}_c(0) \neq 0$, Propositions 4 and 5 imply that $y_c(\infty) - y_c(0) \neq 0$. Therefore, (8) is satisfied with either $t = t_1$ or $t = t_2$. □

The following example shows that the sufficient condition given by Theorem 3 is not necessary.

Example 5

Consider the transfer functions

$$G_c(s) = \frac{s+0.1}{(s+4+\jmath 9)(s+4-\jmath 9)}, \tag{42}$$

$$G_c(s) = \frac{(s-1+\jmath 1)(s-1-\jmath 1)}{(s+2)^3}, \tag{43}$$

and let $x_c(0)=0$. Note that (42) and (43) have no real zeros greater than zero, but the unit step responses $y_c(t)$ of both (42) and (43) have delayed undershoot, as shown in Figure 5. ◊

The next example shows that the presence or absence of delayed undershoot depends on the initial state.

Example 6

Let

$$G_c(s) = \frac{2000(s-1)(s-2)}{(s+2)^3}, \tag{44}$$

which has a minimal realization (A_c, B_c, C_c), where

$$A_c = \begin{bmatrix} -6 & -3 & -1 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 64 \\ 0 \\ 0 \end{bmatrix}, \tag{45}$$

$$C_c = \frac{1}{16}[500 \ \ -375 \ \ 125]. \tag{46}$$

Let $\bar{u}=1$, and consider the response $y_c(t)$ with the initial states $x_c(0)=0$ and $x_c(0)=[0 \ 20 \ 0]^{\mathrm{T}}$. Note that for $x_c(0)=0$, (12) implies that $\tilde{G}_c(s)=G_c(s)$, which has exactly two real zeros greater than zero, whereas for $x_c(0)=[0 \ 20 \ 0]^{\mathrm{T}}$, (12) implies that

$$\tilde{G}_c(s) = \frac{437.5(s+0.29+\jmath 4.2)(s+0.29-\jmath 4.2)}{(s+2)^3}, \tag{47}$$

which has no real zeros greater than zero. Thus, Theorem 3 implies that $y_c(t)$ has delayed undershoot (but not initial undershoot) with $x_c(0)=0$, as shown in Figure 6. For the case where $x_c(0)=[0 \ 20 \ 0]^T$, $\tilde{G}_c(s)$ has no real zeros greater than zero, and Theorem 3 is not applicable. Nevertheless, $y_c(t)$ does not have delayed undershoot with $x_c(0)=[0 \ 20 \ 0]^T$, as shown in Figure 6. ◊

The following result provides a lower bound on the maximum deviation of $y_c(t)$ from $y_c(0)$ in the direction that is opposite to the asymptotic direction. This result generalizes [3, Corollary 1.3.6] to the case where $y_c(0)$ is not necessarily zero and $y_c(\infty)$ is not necessarily one.

Theorem 4

Let $x_c(0)\in\mathbb{R}^n$, define $\tilde{G}_c(s)$ by (12), assume that $\tilde{G}_c(0)\neq 0$, and assume that $\tilde{G}_c(s)$ has at least one real zero greater than zero, namely, z_1. Furthermore, define $\delta_c>0$ by

$$\delta_c \triangleq \begin{cases} y_c(0)-\min_{t\geq 0} y_c(t), & y_c(\infty)>y_c(0), \\ \max_{t\geq 0} y_c(t)-y_c(0), & y_c(\infty)<y_c(0), \end{cases} \tag{48}$$

which is the maximum deviation of $y_c(t)$ from $y_c(0)$ in the direction opposite to the asymptotic direction. Finally, let $\varepsilon_s>0$, and define

$$t_s \triangleq \inf\{t>0: |y_c(t)-y_c(\infty)|<\varepsilon_s\}. \tag{49}$$

Then,

$$\frac{|y_c(\infty)-y_c(0)|-\varepsilon_s}{e^{z_1 t_s}-1} \leq \delta_c. \tag{50}$$

Proof

Proposition 4 implies that

$$\int_0^\infty e^{-z_1 t}[y_c(t)-y_c(0)]\,\mathrm{d}t = \tilde{G}_c(z_1)\frac{\bar{u}}{z_1} = 0, \tag{51}$$

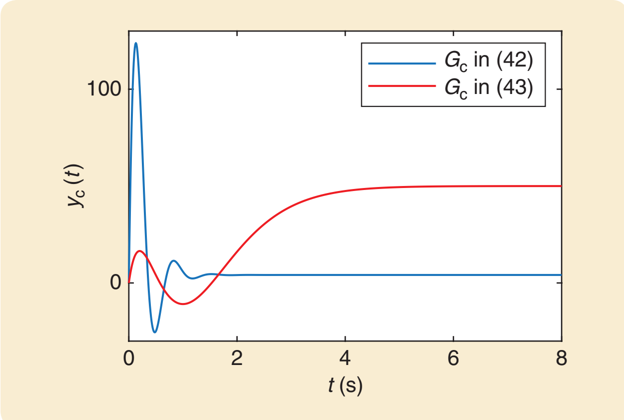

FIGURE 5 Example 5. The unit step responses $y_c(t)$ of (42) and (43). Although (42) and (43) have no real zeros greater than zero, $y_c(t)$ has delayed undershoot for both transfer functions.

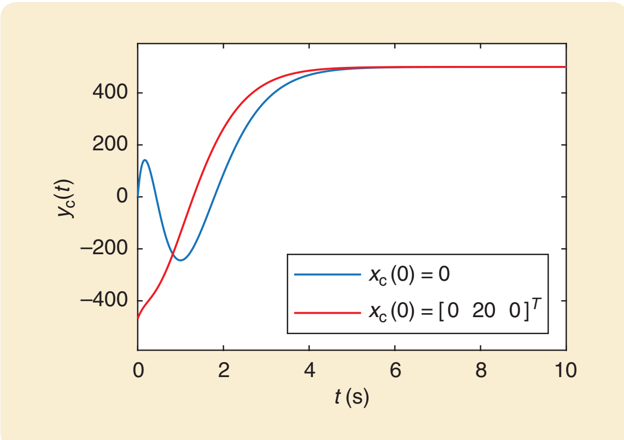

FIGURE 6 Example 6. The response $y_c(t)$ of (44) for two initial states. With $x_c(0)=0$, $\tilde{G}_c(s)=G_c(s)$ has exactly two real zeros greater than zero, and y_c has delayed undershoot. With $x_c(0)=[0\,20\,0]^T$, $\tilde{G}_c(s)$ has no real zeros greater than zero, and $y_c(t)$ does not have delayed undershoot.

where, since $\tilde{G}_c(s)$ is asymptotically stable and $z_1 > 0$, the integral is convergent. Furthermore,

$$\int_0^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt = \int_0^{t_s} e^{-z_1 t}[y_c(t) - y_c(0)]\,dt + \int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt,$$

which, combined with (51), implies that

$$\int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt = -\int_0^{t_s} e^{-z_1 t}[y_c(t) - y_c(0)]\,dt. \quad (52)$$

Since $\tilde{G}_c(0) \neq 0$, Propositions 4 and 5 imply that either $y_c(\infty) > y_c(0)$ or $y_c(\infty) < y_c(0)$. In the case where $y_c(\infty) > y_c(0)$, (52) implies that

$$\int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt \le \delta_c \int_0^{t_s} e^{-z_1 t}\,dt = \frac{\delta_c(1 - e^{-z_1 t_s})}{z_1}. \quad (53)$$

Moreover, since $y_c(t) \ge y_c(\infty) - \varepsilon_s$, for all $t \ge t_s$, it follows that

$$\int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt \ge [y_c(\infty) - y_c(0) - \varepsilon_s]\int_{t_s}^\infty e^{-z_1 t}\,dt = \frac{[y_c(\infty) - y_c(0) - \varepsilon_s]e^{-z_1 t_s}}{z_1}. \quad (54)$$

Combining (53) and (54) yields

$$\frac{y_c(\infty) - y_c(0) - \varepsilon_s}{e^{z_1 t_s} - 1} = \frac{|y_c(\infty) - y_c(0)| - \varepsilon_s}{e^{z_1 t_s} - 1} \le \delta_c.$$

In the case where $y_c(\infty) < y_c(0)$, (52) implies that

$$\int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt \ge -\delta_c \int_0^{t_s} e^{-z_1 t}\,dt = \frac{\delta_c(e^{-z_1 t_s} - 1)}{z_1}. \quad (55)$$

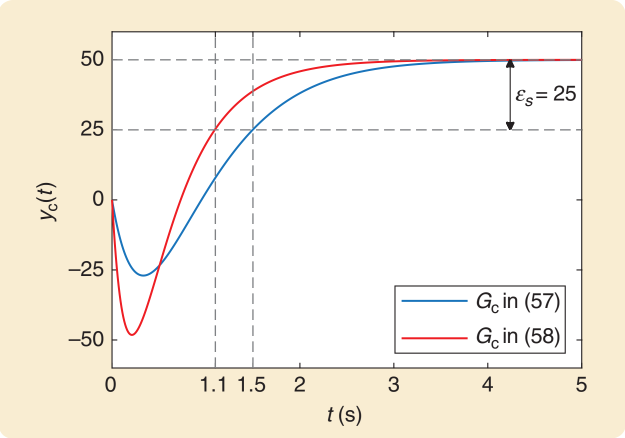

FIGURE 7 Example 7. The unit step responses of (57) and (58), each of which has exactly one real zero at one. Since (57) has slower dynamics than (58), (50) implies that, for a given ε_s, the lower bound for δ_c in the case of (57) is smaller than the lower bound for δ_c in the case of (58).

Moreover, since $y_c(t) \le y_c(\infty) + \varepsilon_s$, for all $t \ge t_s$, it follows that

$$\int_{t_s}^\infty e^{-z_1 t}[y_c(t) - y_c(0)]\,dt \le [y_c(\infty) - y_c(0) + \varepsilon_s]\int_{t_s}^\infty e^{-z_1 t}\,dt = \frac{[y_c(\infty) - y_c(0) + \varepsilon_s]e^{-z_1 t_s}}{z_1}. \quad (56)$$

Combining (55) and (56) yields

$$\frac{y_c(0) - y_c(\infty) - \varepsilon_s}{e^{z_1 t_s} - 1} = \frac{|y_c(\infty) - y_c(0)| - \varepsilon_s}{e^{z_1 t_s} - 1} \le \delta_c. \quad \square$$

In addition to providing a lower bound on δ_c, Theorem 4 implies a relationship between the transient and asymptotic performances of dynamic systems that have delayed undershoot. The following example demonstrates one application of Theorem 4.

Example 7

Consider the transfer functions

$$G_c(s) = \frac{-200(s-1)}{(s+2)^2}, \quad (57)$$

$$G_c(s) = \frac{-600(s-1)}{(s+2)(s+6)}, \quad (58)$$

both of which have exactly one zero at one. Let $\bar{u} = 1$ and $x_c(0) = 0$, and note that, for both (57) and (58), Proposition 5 implies that $y_c(\infty) = 50$. Moreover, note that (57) has slower dynamics than (58). Let $\varepsilon_s = 25$, and define δ_c and t_s by (48) and (49), respectively. Consider the unit step responses of (57) and (58) shown in Figure 7. For (57), $t_s \approx 1.5$ s, which [using (50)] implies that $\delta_c \ge 7.18$. The lower bound on δ_c is conservative by a factor of about 3.8, as shown in Figure 7, where $\delta_c \approx 27$. Similarly, for (58), $t_s \approx 1.1$ s, which [using (50)] implies that $\delta_c \ge 12.47$. The lower bound on δ_c is conservative by a factor of about 3.8, as shown in Figure 7, where $\delta_c \approx 48$. Since (57) has slower dynamics than (58), (50) implies that, for a given ε_s, δ_c has a smaller lower bound for (57) than for (58). ◊

SETPOINT COMMAND RESPONSE WITH INITIAL UNDERSHOOT

In a control system application, a setpoint command is a step input. A setpoint command is specified at start-up and may change during operation. When the setpoint command changes, the closed-loop system has a possibly—and almost always—nonzero internal state due to the internal states of the plant and controller. As shown by Examples 4 and 6, the presence or absence of initial and delayed undershoot depends on the initial state. The setpoint response of the closed-loop system thus depends on the internal state when the setpoint changes; this state may be unknown to the system operator. Figure 8 shows a block diagram of the basic servo loop for the continuous-time system (1) and (2), where $r(t)$ is the command, $e(t) \triangleq r(t) - y_c(t)$ is the error, and $C_c(s)$ is the continuous-time controller.

To demonstrate the effect of a changing setpoint, consider the time-varying step input

$$\bar{u}(t) = \begin{cases} \bar{u}_1 & t \in [0, t_1), \\ \bar{u}_2 & t \in [t_1, t_2), \\ \bar{u}_3 & t \in [t_2, \infty), \end{cases} \tag{59}$$

where $t_2 \gg t_1 \gg 0$ are such that $\dot{x}_c(t_1) \approx 0$ and $\dot{x}_c(t_2) \approx 0$. Setting $t_0 \triangleq 0$ and using (12), define, for $i \in \{1, 2, 3\}$,

$$\tilde{G}_{c,i}(s) \triangleq C_c(sI - A_c)^{-1}\left[\frac{1}{\bar{u}_i}A_c x_c(t_{i-1}) + B_c\right]. \tag{60}$$

Note that

$$\begin{aligned} \tilde{G}_{c,1}(s) &= C_c(sI - A_c)^{-1}\left[\frac{1}{\bar{u}_1}A_c x_c(0) + B_c\right] \\ &= G_c(s) + \frac{1}{\bar{u}_1}C_c(sI - A_c)^{-1}A_c x_c(0). \end{aligned} \tag{61}$$

Moreover, since $\dot{x}_c(t_1) = A_c x_c(t_1) + B_c \bar{u}_1 \approx 0$, it follows that $A_c x_c(t_1) \approx -B_c \bar{u}_1$, and thus,

$$\begin{aligned} \tilde{G}_{c,2}(s) &= C_c(sI - A_c)^{-1}\left[\frac{1}{\bar{u}_2}A_c x_c(t_1) + B_c\right] \\ &\approx C_c(sI - A_c)^{-1}B_c\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right) \\ &= G_c(s)\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right). \end{aligned} \tag{62}$$

Similarly, since $\dot{x}_c(t_2) = A_c x_c(t_2) + B_c \bar{u}_2 \approx 0$, it follows that $A_c x_c(t_2) \approx -B_c \bar{u}_2$, and thus,

$$\begin{aligned} \tilde{G}_{c,3}(s) &= C_c(sI - A_c)^{-1}\left[\frac{1}{\bar{u}_3}A_c x_c(t_2) + B_c\right] \\ &\approx C_c(sI - A_c)^{-1}B_c\left(1 - \frac{\bar{u}_2}{\bar{u}_3}\right) \\ &= G_c(s)\left(1 - \frac{\bar{u}_2}{\bar{u}_3}\right). \end{aligned} \tag{63}$$

Note that $G_{c,2}(s)$ and $G_{c,3}(s)$ have the same zeros as $G_c(s)$. However, the zeros of $G_{c,1}(s)$ may be different from those of $G_c(s)$. The following example illustrates these observations.

Example 8

Let

$$G_c(s) = \frac{-4(s-1)}{(s+2)^2}, \tag{64}$$

which has the minimal realization

$$A_c = \begin{bmatrix} -4 & -2 \\ 2 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C_c = [-2 \;\; 1]. \tag{65}$$

Furthermore, let $\bar{u}_1 = 10$, $\bar{u}_2 = 20$, $\bar{u}_3 = 10$, $t_1 = 10$, and $t_2 = 20$, with the initial condition $x_c(0) = [5 \;\; 5]^T$. Since $G_c(0) = 1$, Proposition 5 implies that $y_c(\infty) = \bar{u}_3$, and, if $t_2 \gg t_1 \gg 0$, then $y_c(t_1) \approx \bar{u}_1$, and $y_c(t_2) \approx \bar{u}_2$. Thus, if $t_2 \gg t_1 \gg 0$, then $\dot{x}_c(t_1) \approx 0$, and $\dot{x}_c(t_2) \approx 0$. Note that (61) yields $\tilde{G}_{c,1}(s) = 3/(s+2)$, which has no real zeros greater than zero. Thus, Theorem 2 implies that, at $t = 0$, $y_c(t)$ does not exhibit initial undershoot, as shown in Figure 9. However, since (62) and (63) imply that $\tilde{G}_{c,2}(s)$ and $\tilde{G}_{c,3}(s)$ have the same zeros as $G_c(s)$, and $G_c(s)$ has one real zero greater than zero, Theorem 2 implies that, at $t = 10$ s and $t = 20$ s, $y_c(t)$ exhibits initial undershoot, as shown in Figure 9. ◇

Example 8 shows that, as long as the internal state of the system converges after each setpoint command, the initial and delayed undershoot are independent of the setpoint command. However, what happens if the setpoint changes *before* the internal state converges? The following example shows that, if the setpoint command changes before the internal state converges, then transitions to the next setpoint at different times may be inconsistent, that is, undershoot may occur in one instance but not in the other. This is enigmatic undershoot.

Example 9

Consider the servo loop shown in Figure 10, where $u(t)$ is the control, $y(t)$ is the measured output, and $e(t) \triangleq r(t) - y(t)$ is the error, where the setpoint command $r(t)$ is given by

$$r(t) = \begin{cases} 10, & 0 \le t < 30, \\ 20, & 30 \le t < 33, \\ 25, & 33 \le t < 60. \end{cases} \tag{66}$$

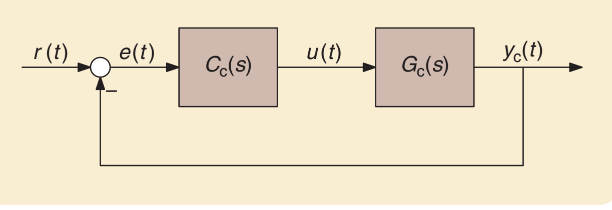

FIGURE 8 The basic servo loop for the continuous-time system (1) and (2).

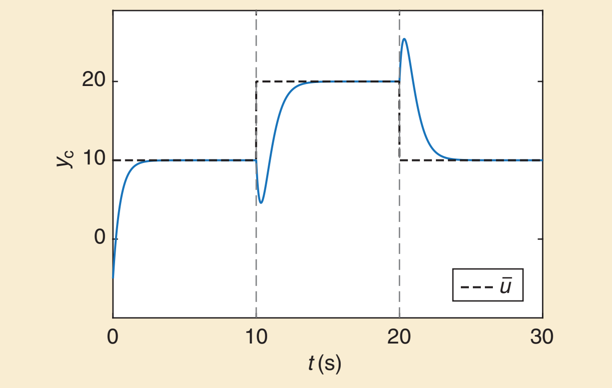

FIGURE 9 Example 8. The response of (64) with the time-varying step input (59). Although (64) has one real zero greater than zero, $y_c(t)$ does not exhibit initial undershoot at $t = 0$ due to the nonzero initial state. However, at both times $t = 10$ s and $t = 20$ s, the state has converged, and, thus, $y_c(t)$ consistently exhibits initial undershoot.

The dynamics are given by the unstable transfer function

$$G_c(s) = \frac{-(s+1)}{(s-2)(s+3)}, \tag{67}$$

which has the minimal realization

$$A_c = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = [-1 \ \ -0.5]. \tag{68}$$

The initial condition is $x_c(0) = 0$. The Matlab commands Kalman and LQI are used to obtain a linear quadratic Gaussian controller that includes an integrator. The transfer function of the controller is given by

$$G_{ue}(s) = \frac{-26.86(s+3)(s+0.07)}{s(s+8.18)(s+0.95)}, \tag{69}$$

which has the minimal realization

$$\bar{A}_c = \begin{bmatrix} -7.88 & -4.30 & -1 \\ -0.45 & -1.22 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{B}_c = \begin{bmatrix} 2.69 \\ 2.45 \\ 1 \end{bmatrix}, \tag{70}$$

$$\bar{C}_c = [-4.19 \ \ -5.96 \ \ -1]. \tag{71}$$

The initial condition is $\bar{x}_c(0) = 0$. The transfer functions $G_{ur}(s)$ and $G_{yr}(s)$ from r to u and from r to y, respectively, are given by

$$G_{ur}(s) = \frac{-26.86(s+0.07)(s-2)(s+3)^2}{(s+3.34)(s+3)(s+2.04)(s+1.58)(s+0.16)}, \tag{72}$$

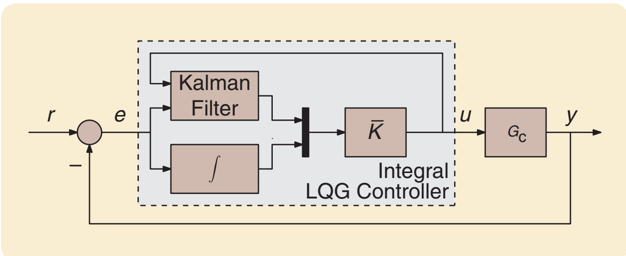

FIGURE 10 Example 9. A servo loop for integral linear quadratic Gaussian (LQG) control of the unstable system (67).

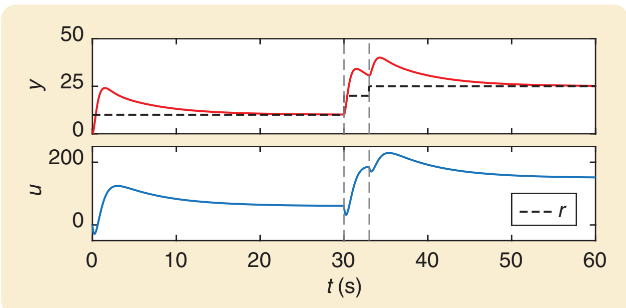

FIGURE 11 Example 9. The setpoint command following with integral linear quadratic Gaussian control for the unstable system (67). At $t = 0$, $u(t)$ exhibits initial undershoot, whereas $y(t)$ does not. Next, at $t = 30$ s, $u(t)$ exhibits initial undershoot, whereas $y(t)$ does not. Finally, at $t = 33$ s, neither $u(t)$ nor $y(t)$ exhibits initial undershoot; however, both $u(t)$ and $y(t)$ have delayed undershoot.

$$G_{yr}(s) = \frac{26.86(s+0.07)(s+1)(s+3)}{(s+3.34)(s+3)(s+2.04)(s+1.58)(s+0.16)}. \tag{73}$$

Note that

$$G_{ur} \sim \left[\begin{array}{cc|c} A_c & B_c\bar{C}_c & 0 \\ -\bar{B}_c C_c & \bar{A}_c & \bar{B}_c \\ \hline 0 & \bar{C}_c & 0 \end{array}\right], \tag{74}$$

$$G_{yr} \sim \left[\begin{array}{cc|c} A_c & B_c\bar{C}_c & 0 \\ -\bar{B}_c C_c & \bar{A}_c & \bar{B}_c \\ \hline C_c & 0 & 0 \end{array}\right], \tag{75}$$

with the internal states $x_{ur}(t) = x_{yr}(t) \triangleq [x_c(t)^T \ \bar{x}_c(t)^T]^T$. Using (12), for $i \in \{1, 2, 3\}$, define

$$\tilde{G}_{ur,i}(s) \triangleq [0 \ \ \bar{C}_c]\left(sI_5 - \begin{bmatrix} A_c & B_c\bar{C}_c \\ -\bar{B}_c C_c & \bar{A}_c \end{bmatrix}\right)^{-1} \times \left(\frac{1}{r_i}\begin{bmatrix} A_c & B_c\bar{C}_c \\ -\bar{B}_c C_c & \bar{A}_c \end{bmatrix} x_{ur,i} + \begin{bmatrix} 0 \\ \bar{B}_c \end{bmatrix}\right),$$

$$\tilde{G}_{yr,i}(s) \triangleq [C_c \ \ 0]\left(sI_5 - \begin{bmatrix} A_c & B_c\bar{C}_c \\ -\bar{B}_c C_c & \bar{A}_c \end{bmatrix}\right)^{-1} \times \left(\frac{1}{r_i}\begin{bmatrix} A_c & B_c\bar{C}_c \\ -\bar{B}_c C_c & \bar{A}_c \end{bmatrix} x_{yr,i} + \begin{bmatrix} 0 \\ \bar{B}_c \end{bmatrix}\right), \tag{76}$$

where

$$r_1 \triangleq 10, \quad r_2 \triangleq 20, \quad r_3 \triangleq 25, \tag{77}$$

$$x_{ur,1} = x_{yr,1} \triangleq 0, \quad x_{ur,2} = x_{yr,2} \triangleq x_{ur}(30), \quad x_{ur,3} = x_{yr,3} \triangleq x_{ur}(33). \tag{78}$$

Since $x_{ur,1} = x_{yr,1} = 0$, it follows that

$$\tilde{G}_{ur,1}(s) = G_{ur}(s), \quad \tilde{G}_{yr,1}(s) = G_{yr}(s). \tag{79}$$

Since, in addition, $G_{ur}(s)$ has exactly one real zero greater than zero, and $G_{yr}(s)$ has no real zeros greater than zero, Theorem 2 implies that, at $t = 0$, $u(t)$ exhibits initial undershoot, but $y(t)$ does not, as shown in Figure 11.

Next, at $t = 30$ s, the setpoint command $r(t)$ changes from 10 to 20. The states of the system and controller at $t = 30$ s are

$$x_c(30) = [0.02 \ \ -20.28]^T, \quad \bar{x}_c(30) = [14 \ \ -5.41 \ \ -87.24]^T, \tag{80}$$

which imply that

$$\tilde{G}_{ur,2}(s) \approx \frac{1}{2}G_{ur}(s), \quad \tilde{G}_{yr,2}(s) \approx \frac{1}{2}G_{yr}(s). \tag{81}$$

Thus, since $G_{ur}(s)$ has exactly one real zero greater than zero, and $G_{yr}(s)$ has no real zeros greater than zero, Theorem 2 implies that, at $t = 30$ s, $u(t)$ exhibits initial undershoot, but $y(t)$ does not, as shown in Figure 11.

Finally, at $t = 33$ s, the setpoint command $r(t)$ changes from 20 to 25. The states of the system and controller at $t = 33$ s are

$$x_c(30) = [-3.61 \ \ -58.26]^T, \quad \bar{x}_c(30) = [23.96 \ \ -28.96 \ \ -105.52]^T, \tag{82}$$

which imply that

$$\tilde{G}_{ur,3}(s) = \frac{-5.37(s-1.73)(s-0.088)(s^2+6s+9)}{(s+3.34)(s+3)(s+2.04)(s+1.58)(s+0.16)}, \quad (83)$$

$$\tilde{G}_{yr,3}(s) = \frac{-0.09(s-50.77)(s+3)(s+1)(s-0.08)}{(s+3.34)(s+3)(s+2.04)(s+1.58)(s+0.16)}. \quad (84)$$

Thus, since both $\tilde{G}_{ur,3}(s)$ and $\tilde{G}_{yr,3}(s)$ have exactly two real zeros greater than zero, Theorem 2 implies that, at $t = 33$ s, neither $u(t)$ nor $y(t)$ exhibits initial undershoot; however, in this case, Theorem 3 implies that both $u(t)$ and $y(t)$ have delayed undershoot, as shown in Figure 11. ◊

PRELIMINARIES FOR DISCRETE-TIME SYSTEMS

Consider the discrete-time system

$$x_{\mathrm{d},k+1} = A_{\mathrm{d}} x_{\mathrm{d},k} + B_{\mathrm{d}} \bar{u}, \quad (85)$$

$$y_{\mathrm{d},k} = C_{\mathrm{d}} x_{\mathrm{d},k} + E_{\mathrm{d}} \bar{u}, \quad (86)$$

where, for all $k \ge 0$, $x_{\mathrm{d},k} \in \mathbb{R}^n$ is the state, $y_{\mathrm{d},k} \in \mathbb{R}$ is the output, and $\bar{u} \in \mathbb{R}$ is a nonzero step input (Table 3). The matrices $(A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}}, E_{\mathrm{d}})$ are assumed to be a minimal state-space realization of the transfer function

$$G_{\mathrm{d}}(z) \triangleq C_{\mathrm{d}}(zI - A_{\mathrm{d}})^{-1} B_{\mathrm{d}} + E_{\mathrm{d}}. \quad (87)$$

Note that $G_{\mathrm{d}}(\infty) = E_{\mathrm{d}}$.

Throughout this article, $G_{\mathrm{d}}(z)$ is assumed to be asymptotically stable, SISO, and of the form

$$G_{\mathrm{d}}(z) = K\frac{N_{\mathrm{d}}(z)}{D_{\mathrm{d}}(z)} = K\frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}, \quad (88)$$

where $n \ge 1$, $n \ge m \ge 0$, $K \ne 0$, and $z_1, \ldots, z_m \in \mathbb{C}$ and $p_1, \ldots, p_n \in \mathbb{C}$ are the zeros and poles of $G_{\mathrm{d}}(z)$, respectively. Since $(A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}})$ is controllable and observable, the polynomials N_{d} and D_{d} have no common roots. Let $d \triangleq n - m \ge 0$ denote the relative degree of $G_{\mathrm{d}}(z)$. Note that $G_{\mathrm{d}}(z)$ is strictly proper, that is, $d > 0$, if and only if $E_{\mathrm{d}} = 0$, and $G_{\mathrm{d}}(z)$ is exactly proper, that is, $d = 0$, if and only if $E_{\mathrm{d}} = K$. If $G_{\mathrm{d}}(z)$ is exactly proper, then $\hat{G}_{\mathrm{d}}(z) \triangleq G_{\mathrm{d}}(z) - E_{\mathrm{d}}$ is the *strictly proper part* of $G_{\mathrm{d}}(z)$, which is given by

$$\hat{G}_{\mathrm{d}}(z) = K\frac{N_{\mathrm{d}}(z)}{D_{\mathrm{d}}(z)} - K = K\frac{N_{\mathrm{d}}(z) - D_{\mathrm{d}}(z)}{D_{\mathrm{d}}(z)}. \quad (89)$$

Since N_{d} and D_{d} have no common roots, it follows that $N_{\mathrm{d}} - D_{\mathrm{d}}$ and D_{d} have no common roots, and, thus, there is no pole-zero cancellation in (89). Furthermore, $N_{\mathrm{d}} - D_{\mathrm{d}}$ and N_{d} have no common roots, and thus, the zeros of $\hat{G}_{\mathrm{d}}(z)$ are different from the zeros of $G_{\mathrm{d}}(z)$. If $G_{\mathrm{d}}(z)$ is strictly proper, then $\hat{G}_{\mathrm{d}}(z) = G_{\mathrm{d}}(z)$.

For all $k \ge 0$, the step response of (85) and (86) in the presence or absence of the possibly nonzero initial state $x_{\mathrm{d},0}$ is given by

$$y_{\mathrm{d},k} \triangleq C_{\mathrm{d}} A_{\mathrm{d}}^k x_{\mathrm{d},0} + \sum_{i=0}^{k-1} C_{\mathrm{d}} A_{\mathrm{d}}^{k-1-i} B_{\mathrm{d}} \bar{u} + E_{\mathrm{d}} \bar{u}. \quad (90)$$

Hence,

$$y_{\mathrm{d},0} = C_{\mathrm{d}} x_{\mathrm{d},0} + E_{\mathrm{d}} \bar{u}. \quad (91)$$

If $x_{\mathrm{d},0} = 0$ and $E_{\mathrm{d}} = 0$, then $y_{\mathrm{d},0} = 0$.

Definition 3

The response $y_{\mathrm{d},k}$ given by (90) has *initial undershoot* if there exists $k_1 \ge 1$ such that, for all $k \in [0, k_1 - 1]$,

$$y_{\mathrm{d},k} = y_{\mathrm{d},0}, \quad (92)$$

and

$$(y_{\mathrm{d},k_1} - y_{\mathrm{d},0})(y_{\mathrm{d},\infty} - y_{\mathrm{d},0}) \le 0. \quad (93)$$

Definition 4

The response $y_{\mathrm{d},k}$ given by (90) has *delayed undershoot* if there exists $k_1 \ge 1$ such that (93) holds.

Note that, if $y_{\mathrm{d},k}$ has initial undershoot, then $y_{\mathrm{d},k}$ has delayed undershoot. However, Example 14 shows that

TABLE 3 A summary of discrete-time and sampled-data definitions and results.

Definition 3	Initial undershoot
Definition 4	Delayed undershoot
Proposition 9	Initial undershoot with the zero initial state
Proposition 10	Shift-invariance property of initial and delayed undershoot
Proposition 11	Application of Proposition 9 for exactly proper systems
Proposition 12	The z-transform of $y_{\mathrm{d},k} - y_{\mathrm{d},0}$
Proposition 13	Asymptotic value $y_{\mathrm{d},\infty}$ of $y_{\mathrm{d},k}$
Proposition 14	First nonzero value of $y_{\mathrm{d},k}$
Proposition 15	Initial undershoot with the zero initial state using the relative degree
Theorem 5	Initial undershoot with the zero initial state using zeros
Theorem 6	Initial undershoot with a nonzero initial state
Theorem 7	Delayed undershoot
Theorem 8	Lower bound on the maximum deviation of $y_{\mathrm{d},k}$ from $y_{\mathrm{d},0}$
Theorem 9	Initial undershoot under sufficiently fast sampling
Theorem 10	Initial undershoot under sufficiently slow sampling

the converse is not true. The special case $y_{d,0}=0$ is worth noting.

Proposition 9

Assume that $y_{d,0}=0$. Then, $y_{d,k}$ has initial undershoot if and only if there exists $k_1 \geq 1$ such that, for all $k \in [0,k_1]$,

$$y_{d,k}=0, \tag{94}$$

and

$$y_{d,k_1} y_{d,\infty} \leq 0. \tag{95}$$

Furthermore, $y_{d,k}$ has delayed undershoot if and only if there exists $k_1 \geq 1$ such that (94) holds.

The following observation is immediate but worth noting. The proof is similar to the proof of Proposition 2.

Proposition 10

Let $\alpha \in \mathbb{R}$. Then, $y_{d,k}$ has initial undershoot if and only if $y_{d,k}-\alpha$ has initial undershoot. In addition, $y_{d,k}$ has delayed undershoot if and only if $y_{d,k}-\alpha$ has delayed undershoot.

Proposition 10 shows that initial undershoot is preserved under an arbitrary, constant offset of the step response. The following result views $E_d\bar{u}$ as a constant offset of the step response of the strictly proper part of $G_d(z)$.

Proposition 11

Let $x_{d,0} \in \mathbb{R}^n$. Then, $y_{d,k}$ has initial undershoot if and only if $y_{d,k}-E_d\bar{u}$ has initial undershoot. In addition, $y_{d,k}$ has delayed undershoot if and only if $y_{d,k}-E_d\bar{u}$ has delayed undershoot.

In view of (90), Proposition 11 shows that, for all initial states $x_{d,0}$, the presence or absence of initial undershoot is independent of the value of E_d. Hence, there is no loss of generality by setting $E_d=0$ in (86), that is, by replacing the exactly proper transfer function $G_d(z)$ with its strictly proper part $\hat{G}_d(z)$. This observation is further justified by the following result, which shows that the z-transform of $y_{d,k}-y_{d,0}$ does not depend on E_d.

Proposition 12

Let $y_{d,k}$ be the step response of (85) and (86) given by (90). Then,

$$\mathcal{Z}\{y_{d,k}-y_{d,0}\} = \tilde{G}_d(z)\frac{z\bar{u}}{z-1}, \tag{96}$$

where

$$\tilde{G}_d(z) \triangleq C_d(zI-A_d)^{-1}\Big[\frac{1}{\bar{u}}(A_d+I)x_{d,0}+B_d\Big]. \tag{97}$$

Proof

Subtracting (91) from (90) and taking the z-transform yields

$$\begin{aligned}
\mathcal{Z}\{y_{d,k}-y_{d,0}\} &= zC_d(zI-A_d)^{-1}x_{d,0}+C_d(zI-A_d)^{-1} \\
&\quad B_d\frac{z\bar{u}}{z-1}-\frac{z}{z-1}C_dx_{d,0} \\
&= zC_d(zI-A_d)^{-1}\Big[x_{d,0}+B_d\frac{\bar{u}}{z-1}\Big]-\frac{z}{z-1}C_dx_{d,0} \\
&= zC_d(zI-A_d)^{-1}\Big[x_{d,0}+B_d\frac{\bar{u}}{z-1} \\
&\quad -\frac{1}{z-1}(zI-A_d)x_{d,0}\Big] \\
&= zC_d(zI-A_d)^{-1}\Big[B_d\frac{\bar{u}}{z-1}+\frac{1}{z-1}(A_d+I)x_{d,0}\Big] \\
&= \tilde{G}_d(z)\frac{z\bar{u}}{z-1}. \qquad \square
\end{aligned}$$

The situation is different, however, for the case of the nonzero initial state since, as shown by (90), the effect of $x_{d,0}$ is not equivalent to a constant offset of $y_{d,k}$. Examples 12 and 14 shows that initial undershoot or delayed undershoot may occur for some initial states but not others.

The following result, which follows from the discrete-time final value theorem [20, pp. 139], provides an expression for the asymptotic value $y_{d,\infty}$ of $y_{d,k}$.

Proposition 13

$y_{d,\infty} \triangleq \lim_{k\to\infty} y_{d,k}$ exists and is given by

$$y_{d,\infty} = \lim_{z\to 1} G_d(z)\bar{u} = G_d(1)\bar{u} = K\frac{N_d(1)}{D_d(1)}\bar{u}. \tag{98}$$

In view of (93), Proposition 13 implies that if

$$G_d(1)\bar{u} = C_dx_{d,0}+E_d\bar{u}, \tag{99}$$

then $y_{d,k}$ has initial undershoot. In particular, in the special case where $x_{d,0}=0$, Proposition 13 implies that if

$$G_d(1)=E_d, \tag{100}$$

then $y_{d,k}$ has initial undershoot. However, Example 12 shows that the converses of these statements are not true.

INITIAL UNDERSHOOT FOR DISCRETE-TIME SYSTEMS WITH ZERO INITIAL STATE

For the case of the zero initial state, this section provides a necessary and sufficient condition for initial undershoot in terms of the zeros of the strictly proper part $\hat{G}_d(z)$ of $G_d(z)$. The following section considers the case of a nonzero initial state.

The following result, which follows from the initial value theorem [25, p. 119], concerns the first nonzero value of the step response $y_{d,k}$.

Proposition 14

Assume that $G_d(z)$ is strictly proper and $x_{d,0}=0$. Then, for all $i \in \{1,\ldots,d-1\}$, $y_{d,i}=\lim_{z\to\infty} z^iG_d(z)\bar{u}=0$. Furthermore, $y_{d,d}=\lim_{z\to\infty} z^dG_d(z)\bar{u}=K\bar{u}$.

Proposition 15
Assume that $x_{\mathrm{d},0}=0$, and let $\hat{d}\geq 1$ denote the relative degree of $\hat{G}_{\mathrm{d}}(z)$. Then, the unit step response $y_{\mathrm{d},k}$ of $G_{\mathrm{d}}(z)$ has initial undershoot if and only if

$$(y_{\mathrm{d},\hat{d}}-E_{\mathrm{d}}\bar{u})(y_{\mathrm{d},\infty}-E_{\mathrm{d}}\bar{u})\leq 0. \tag{101}$$

Proof
Since $y_{\mathrm{d},k}-E_{\mathrm{d}}\bar{u}$ is the step response of $\hat{G}_{\mathrm{d}}(z)$, Propositions 9 and 14 imply that $y_{\mathrm{d},k}-E_{\mathrm{d}}\bar{u}$ has initial undershoot if and only if (101) is satisfied. Thus, $y_{\mathrm{d},k}$ has initial undershoot if and only if (101) is satisfied. □

Theorem 5
Assume that $x_{\mathrm{d},0}=0$ and $G_{\mathrm{d}}(1)\neq E_{\mathrm{d}}$. Then, the step response $y_{\mathrm{d},k}$ of $G_{\mathrm{d}}(z)$ has initial undershoot if and only if $\hat{G}_{\mathrm{d}}(z)$ has an odd number of real zeros greater than one.

Proof
First, consider the case where $E_{\mathrm{d}}=0$, that is, $G_{\mathrm{d}}(z)$ is strictly proper. Thus, Propositions 13 and 14 imply that

$$y_{\mathrm{d},d}y_{\mathrm{d},\infty}=K^2\bar{u}^2\frac{N_{\mathrm{d}}(1)}{D_{\mathrm{d}}(1)}. \tag{102}$$

Since $G_{\mathrm{d}}(1)\neq 0$, it follows that $N_{\mathrm{d}}(1)\neq 0$. Moreover, since $G_{\mathrm{d}}(z)$ is asymptotically stable, every root of $D_{\mathrm{d}}(z)$ is contained in the open unit disk. Therefore, using Lemma S1, it follows from (102) that

$$\begin{aligned}\operatorname{sign}(y_{\mathrm{d},d}y_{\mathrm{d},\infty})&=\operatorname{sign}\left(K^2\bar{u}^2\frac{N_{\mathrm{d}}(1)}{D_{\mathrm{d}}(1)}\right)=\operatorname{sign}\frac{(-1)^{\pi_{N_{\mathrm{d}}}(1)}}{(-1)^{\pi_{D_{\mathrm{d}}}(1)}}\\&=\operatorname{sign}\frac{(-1)^{\pi_{N_{\mathrm{d}}}(1)}}{(-1)^0}=(-1)^{\pi_{N_{\mathrm{d}}}(1)},\end{aligned}$$

where $\pi_p(\alpha)$ is the number, counting multiplicity, of real roots greater than $\alpha\in\mathbb{R}$ of the polynomial p. Thus, $y_{\mathrm{d},k}$ has initial undershoot if and only if $\pi_{N_{\mathrm{d}}}(1)$ is odd. Therefore, Proposition 15 implies that $y_{\mathrm{d},k}$ has initial undershoot if and only if $\pi_{N_{\mathrm{d}}}(1)$ is odd, that is, if and only if $G_{\mathrm{d}}(z)=\hat{G}_{\mathrm{d}}(z)$ has an odd number of real zeros greater than one.

Next, consider the case where $E_{\mathrm{d}}\neq 0$. Since $y_{\mathrm{d},k}-E_{\mathrm{d}}\bar{u}$ is the step response of the strictly proper transfer function $\hat{G}_{\mathrm{d}}(z)$, it follows that $y_{\mathrm{d},k}-E_{\mathrm{d}}\bar{u}$ has initial undershoot if and only if $\hat{G}_{\mathrm{d}}(z)$ has an odd number of real zeros greater than one. Thus, Proposition 11 implies that $y_{\mathrm{d},k}$ has initial undershoot if and only if $\hat{G}_{\mathrm{d}}(z)$ has an odd number of real zeros greater than one. □

The following example (Table 4) illustrates Propositions 13–15 and Theorem 5.

Example 10
Let $x_{\mathrm{d},0}=0$ and $\bar{u}=1$, and consider the transfer function

$$G_{\mathrm{d}}(z)=\frac{-3(z-2)}{(z-0.1)(z-0.2)(z-0.3)(z-0.4)}, \tag{103}$$

which has exactly one real zero greater than one. Hence, Theorem 5 implies that $y_{\mathrm{d},k}$ has initial undershoot.

To show this more directly, Proposition 14 implies that

$$y_{\mathrm{d},1}=\lim_{z\to\infty}zG_{\mathrm{d}}(z)=0, \tag{104}$$

$$y_{\mathrm{d},2}=\lim_{z\to\infty}z^2G_{\mathrm{d}}(z)=0, \tag{105}$$

$$y_{\mathrm{d},3}=\lim_{z\to\infty}z^3G_{\mathrm{d}}(z)=-3. \tag{106}$$

Alternatively,

$$\begin{aligned}y_{\mathrm{d},k}&=\mathcal{Z}^{-1}\left\{\frac{G_{\mathrm{d}}(z)z}{z-1}\right\}\\&=\mathcal{Z}^{-1}\left\{\frac{10}{z-1}+\frac{105.6}{z-0.1}-\frac{675}{z-0.2}+\frac{1092.9}{z-0.3}-\frac{533.3}{z-0.4}\right\}\\&=\begin{cases}0, & k=0,\\ 10+105.6(0.1^{k-1})-675(0.2^{k-1})\\ \quad+1092.9(0.3^{k-1})-533.3(0.4^{k-1}), & k\geq 1.\end{cases}\end{aligned} \tag{107}$$

It follows from (107) that $y_{\mathrm{d},1}=y_{\mathrm{d},2}=0$, and $y_{\mathrm{d},3}=-3$, as shown in (104)–(106). Figure 12 shows the unit step response of (103). Note that Proposition 13 yields $\lim_{k\to\infty}y_{\mathrm{d},k}=\lim_{z\to 1}G_{\mathrm{d}}(z)=G_{\mathrm{d}}(1)=9.92$, as shown in Figure 12. Thus, since $y_{\mathrm{d},d}y_{\mathrm{d},\infty}<0$, Proposition 15 implies that $y_{\mathrm{d},k}$ has initial undershoot. ◊

In the case where $x_{\mathrm{d},0}=0$, Theorem 5 provides necessary and sufficient conditions for the existence of initial undershoot. Although $G_{\mathrm{d}}(z)$ may be either strictly proper or exactly proper, this necessary and sufficient condition concerns the strictly proper part $\hat{G}_{\mathrm{d}}(z)$ of $G_{\mathrm{d}}(z)$. In the case where $G_{\mathrm{d}}(z)$ is exactly proper, $G_{\mathrm{d}}(z)$ has n zeros, and $\hat{G}_{\mathrm{d}}(z)$ has $m<n$ zeros. This leads to the question as to whether or not the presence or absence of initial undershoot can be directly characterized in terms of the zeros of $G_{\mathrm{d}}(z)$ rather than indirectly in terms of the zeros of $\hat{G}_{\mathrm{d}}(z)$. The following example investigates this question.

Example 11
Let E_{d} be a nonzero real number, and let $G_{\mathrm{d}}(z)$ be the transfer function

TABLE 4 A summary of discrete-time and sampled-data examples.

Example 10	Initial undershoot with the zero initial state
Example 11	Zeros of exactly proper systems
Example 12	Initial undershoot with a nonzero initial state
Example 13	Delayed undershoot
Example 14	Delayed undershoot
Example 15	Application of Theorem 4
Example 16	Initial undershoot for a sampled-data system
Example 17	Initial undershoot for a sampled-data system
Example 18	Initial undershoot for a sampled-data system

$$G_{\rm d}(z) = \frac{z-2}{z^2} + E_{\rm d} = \frac{E_{\rm d}z^2+z-2}{z^2}, \tag{108}$$

whose strictly proper part has exactly one real zero greater than one. Furthermore,

$$G_{\rm d}(z) = \frac{(z-z_1)(z-z_2)}{z^2}, \tag{109}$$

where, for $E_{\rm d} \neq 0$,

$$z_1 = \frac{-1+\sqrt{1+8E_{\rm d}}}{2E_{\rm d}}, \tag{110}$$

$$z_2 = \frac{-1-\sqrt{1+8E_{\rm d}}}{2E_{\rm d}}. \tag{111}$$

Therefore, if $E_{\rm d} \in (-\infty, -1/8)$, then z_1 and z_2 are complex conjugates, whereas if $E_{\rm d} = -1/8$, then $z_1 = z_2 = 4$. Furthermore, as $E_{\rm d}$ increases from $-1/8$ to zero, z_1 and z_2 depart from four in opposite directions, with $\lim_{E_{\rm d}\uparrow 0} z_1 = 2$, and $\lim_{E_{\rm d}\uparrow 0} z_2 = \infty$. Finally, as $E_{\rm d}$ increases from zero to ∞, z_1 decreases from two to zero, and z_2 increases from $-\infty$ to zero. Therefore, although the strictly proper part of $G_{\rm d}(z)$ has exactly one real zero greater than one, $G_{\rm d}(z)$ may have zero real zeros, exactly two real zeros less than one, exactly one real zero greater than one, or exactly two real zeros greater than one. Therefore, the zeros of an exactly proper transfer function cannot be used to determine the presence or absence of initial undershoot. ◊

INITIAL UNDERSHOOT FOR DISCRETE-TIME SYSTEMS WITH NONZERO INITIAL STATE

For the case where the initial state is not necessarily zero, this section provides a necessary and sufficient condition for initial undershoot in terms of the zeros of $\tilde{G}_{\rm d}(z)$ defined by (97).

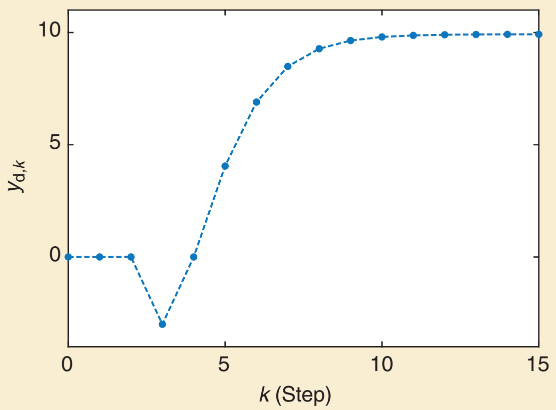

FIGURE 12 Example 10. The unit step response of (103). Note that $y_{{\rm d},1} = y_{{\rm d},2} = 0$. However, since $y_{{\rm d},3} < 0$, and the asymptotic response is $y_{{\rm d},\infty} = 9.92 > 0$, it follows that $y_{{\rm d},k}$ has initial undershoot. Alternatively, since $G_{\rm d}(z)$ has exactly one real zero greater than one, $y_{{\rm d},k}$ has initial undershoot. Note that the dotted lines are provided only to help visualize the initial undershoot.

The following result is a consequence of Theorem 5 and Propositions 10 and 12.

Theorem 6

Assume that $x_{{\rm d},0} \in \mathbb{R}^n$ and $\tilde{G}_{\rm d}(1) \neq 0$, where $\tilde{G}_{\rm d}(z)$ is given by (97). Then, the step response $y_{{\rm d},k}$ has initial undershoot if and only if $\tilde{G}_{\rm d}(z)$ has an odd number of real zeros greater than one.

Proof

For the realization $(A_{\rm d}, (1/\bar{u})(A_{\rm d}+I)x_{{\rm d},0}+B_{\rm d}, C_{\rm d})$ of $\tilde{G}_{\rm d}(z)$, Proposition 12 implies that

$$\tilde{x}_{{\rm d},k+1} = A_{\rm d}\tilde{x}_{{\rm d},k} + \Big[\frac{1}{\bar{u}}(A_{\rm d}+I)x_{{\rm d},0}+B_{\rm d}\Big]\bar{u},$$

$$y_{{\rm d},k} - y_{{\rm d},0} = C_{\rm d}\tilde{x}_{{\rm d},k},$$

where the internal state $\tilde{x}_{{\rm d},k} \in \mathbb{R}^n$ satisfies $\tilde{x}_{{\rm d},0} = 0$. Thus, since $\tilde{G}_{\rm d}(z)$ is strictly proper and $\tilde{G}_{\rm d}(1) \neq 0$, Theorem 5 implies that $y_{{\rm d},k} - y_{{\rm d},0}$ has initial undershoot if and only if $\tilde{G}_{\rm d}(z)$ has an odd number of real zeros greater than one. Therefore, Proposition 10 implies that $y_{{\rm d},k}$ has initial undershoot if and only if $\tilde{G}_{\rm d}(z)$ has an odd number of real zeros greater than one. □

Note that, if $x_{{\rm d},0} = 0$, then (97) implies that $\tilde{G}_{\rm d}(z) = \hat{G}_{\rm d}(z)$; otherwise, $\tilde{G}_{\rm d}(z)$ and $\hat{G}_{\rm d}(z)$ may have different zeros, as demonstrated by the following example.

Example 12

Let

$$G_{\rm d}(z) = \frac{200(z-4)(z-1.5)(z-0.4)}{(z-0.3)^3}, \tag{112}$$

which has a minimal realization $(A_{\rm d}, B_{\rm d}, C_{\rm d}, 200)$, where

$$A_{\rm d} = \frac{1}{1000}\begin{bmatrix} 900 & -540 & 216 \\ 500 & 0 & 0 \\ 0 & 250 & 0 \end{bmatrix}, \quad B_{\rm d} = \begin{bmatrix} 64 \\ 0 \\ 0 \end{bmatrix}, \tag{113}$$

$$C_{\rm d} = \frac{1}{80}[-1250 \ \ 3965 \ \ -4746]. \tag{114}$$

Note that

$$\hat{G}_{\rm d}(z) = G_{\rm d}(z) - 200 = \frac{-1000(z-1.19)(z-0.4)}{(z-0.3)^3}, \tag{115}$$

and, thus, $\hat{d} = 1$. Let $\bar{u} = 1$, and consider the step response $y_{{\rm d},k}$ of $G_{\rm d}(z)$ with initial states $x_{{\rm d},0} = 0$ and $x_{{\rm d},0} = [0 - 25 \ \ 0]^T$. Note that, for $x_{{\rm d},0} = 0$, (97) implies that $\tilde{G}_{\rm d}(z) = \hat{G}_{\rm d}(z)$. Thus, for $x_{{\rm d},0} = 0$, $\hat{G}_{\rm d}(z)$ has exactly one real zero greater than one, whereas for $x_{{\rm d},0} = [0 - 25 \ \ 0]^T$, (97) implies that

$$\tilde{G}_{\rm d}(z) = \frac{-2079.2(z-0.99)(z-0.40)}{(z-0.3)^3}, \tag{116}$$

which has no real zeros greater than one. Thus, Theorem 6 implies that $y_{{\rm d},k}$ has initial undershoot with $x_{{\rm d},0} = 0$ but does not have initial undershoot with $x_{d,0} = [0 - 25 \;\; 0]^T$, as shown in Figure 13. ◊

DELAYED UNDERSHOOT IN DISCRETE-TIME SYSTEMS

For the case where the initial state is not necessarily zero, this section provides a sufficient condition for delayed undershoot in terms of the zeros of $\tilde{G}_d(z)$ defined by (97).

Theorem 7

Let $x_{d,0} \in \mathbb{R}^n$, define $\tilde{G}_d(z)$ by (97), assume that $\tilde{G}_d(1) \neq 0$, and assume that $\tilde{G}_d(z)$ has at least one real zero greater than one. Then, the response $y_{d,k}$ has delayed undershoot.

Proof

Let $z_1 \in (1,\infty)$ be a real zero of $\tilde{G}_d(z)$. Thus, Proposition 12 implies that

$$\sum_{k=0}^{\infty} z_1^{-k}(y_{d,k} - y_{d,0}) = \frac{z_1}{z_1 - 1}\tilde{G}_d(z_1) = 0, \qquad (117)$$

where, since $\tilde{G}_d(z)$ is asymptotically stable, and $z_1 > 1$, the sum is convergent. Since $z_1^{-k} > 0$, for all $k \geq 0$, (117) implies that there exist $k_2 > 0$ and $k_3 > 0$ such that $y_{d,k_2} - y_{d,0} > 0$ and $y_{d,k_3} - y_{d,0} < 0$. Moreover, since $\tilde{G}_d(1) \neq 0$, Propositions 13 and 12 imply that $y_{d,\infty} - y_{d,0} \neq 0$. Therefore, (93) is satisfied with either $k_1 = k_2$ or $k_1 = k_3$. □

The following example shows that the sufficient condition given by Theorem 7 is not necessary.

Example 13

Consider the transfer functions

$$G_d(z) = \frac{30(z-0.2)}{(z+0.5+\jmath 0.5)(z+0.5+\jmath 0.5)}, \qquad (118)$$

$$G_d(z) = \frac{4(z-2+\jmath)(z-2-\jmath)}{(z-0.4)^3}, \qquad (119)$$

and let $x_{d,0} = 0$. Note that (118) and (119) have no real zeros greater than one, but the unit step response $y_{d,k}$ of both (118) and (119) has delayed undershoot, as shown in Figure 14. ◊

The next example shows that the presence or absence of delayed undershoot depends on the initial state.

Example 14

Let

$$G_d(z) = \frac{1000(z-1.1)(z-2.2)}{(z-0.3)^3}, \qquad (120)$$

which has a minimal realization $(A_d, B_d, C_d, 0)$, where

$$A_d = \frac{1}{1000}\begin{bmatrix} 900 & -540 & 216 \\ 500 & 0 & 0 \\ 0 & 250 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 128 \\ 0 \\ 0 \end{bmatrix}, \qquad (121)$$

$$C_d = \frac{1}{16}[125 \;\; -825 \;\; 2420]. \qquad (122)$$

Let $\bar{u} = 1$, and consider the response $y_{d,k}$ with initial states $x_{d,0} = 0$ and $x_{d,0} = [30 \;\; -30 \;\; -30]^T$. Note that for $x_{d,0} = 0$, (97) implies that $\tilde{G}_d(z) = G_d(z)$, which has exactly two real zeros greater than one. For $x_{d,0} = [30 \;\; -30 \;\; -30]^T$, (97) implies that

$$\tilde{G}_d(z) = \frac{3377.2(z+1.11)(z-0.76)}{(z-0.3)^3}, \qquad (123)$$

which has no real zeros greater than one. Thus, Theorem 7 implies that $y_{d,k}$ has delayed undershoot with $x_{d,0} = 0$, as shown in Figure 15. Note that for the case where $x_{d,0} = [30 \;\; -30 \;\; -30]^T$, $\tilde{G}_d(z)$ has no real zeros greater

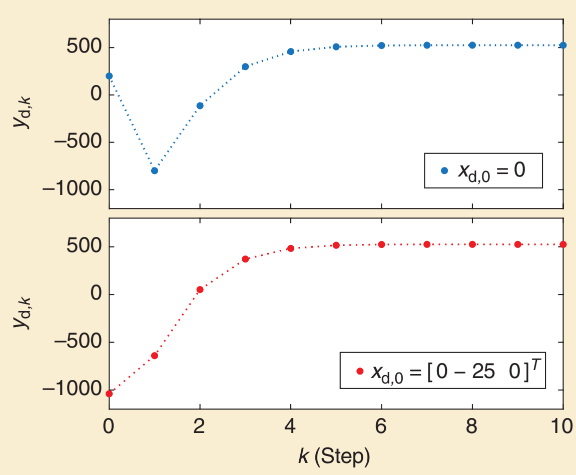

FIGURE 13 Example 12. The response $y_{d,k}$ of (112) for two initial states. With $x_{d,0} = 0$, $\tilde{G}_d(z) = \hat{G}_d(z)$ has exactly one real zero greater than one, and, thus, $y_{d,k}$ has initial undershoot, whereas with $x_{d,0} = [0 - 25 \;\; 0]^T$, $\tilde{G}_d(z)$ has no real zeros greater than one, and thus, $y_{d,k}$ does not have initial undershoot.

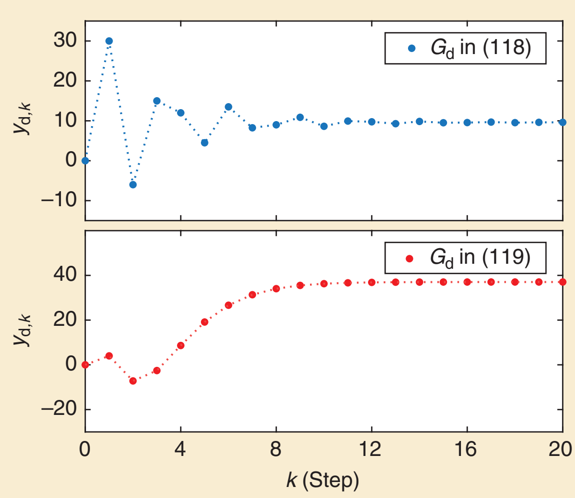

FIGURE 14 Example 13. The unit step response $y_{d,k}$ of (118) and (119). Although (118) and (119) have no real zeros greater than one, $y_{d,k}$ has delayed undershoot.

than zero, and Theorem 7 is not applicable. Nevertheless, in this case, $y_{\mathrm{d},k}$ does not have delayed undershoot with $x_{\mathrm{d},0} = [30\ -30\ -30]^T$, as shown in Figure 15. ◊

The following result provides a lower bound on the maximum deviation of $y_{\mathrm{d},k}$ from $y_{\mathrm{d},0}$ in the direction that is opposite to the asymptotic direction.

Theorem 8

Let $x_{\mathrm{d},0} \in \mathbb{R}^n$, define $\tilde{G}_{\mathrm{d}}(z)$ by (97), assume that $\tilde{G}_{\mathrm{d}}(1) \neq 0$, and assume that $\tilde{G}_{\mathrm{d}}(z)$ has at least one real zero greater than one, namely z_1. Furthermore, define $\delta_{\mathrm{d}} > 0$ by

$$\delta_{\mathrm{d}} \triangleq \begin{cases} y_{\mathrm{d},0} - \min_{k \geq 0} y_{\mathrm{d},k}, & y_{\mathrm{d},\infty} > y_{\mathrm{d},0}, \\ \max_{k \geq 0} y_{\mathrm{d},k} - y_{\mathrm{d},0}, & y_{\mathrm{d},\infty} < y_{\mathrm{d},0}, \end{cases} \tag{124}$$

which is the maximum deviation of $y_{\mathrm{d},k}$ from $y_{\mathrm{d},0}$ in the direction that is opposite to the asymptotic direction. Finally, let $\varepsilon_s > 0$ and define

$$k_s \triangleq \min\{k > 0\colon |y_{\mathrm{d},k} - y_{\mathrm{d},\infty}| < \varepsilon_s\}. \tag{125}$$

Then,

$$\frac{|y_{\mathrm{d},\infty} - y_{\mathrm{d},0}| - \varepsilon_s}{z_1^{k_s} - 1} \leq \delta_{\mathrm{d}}. \tag{126}$$

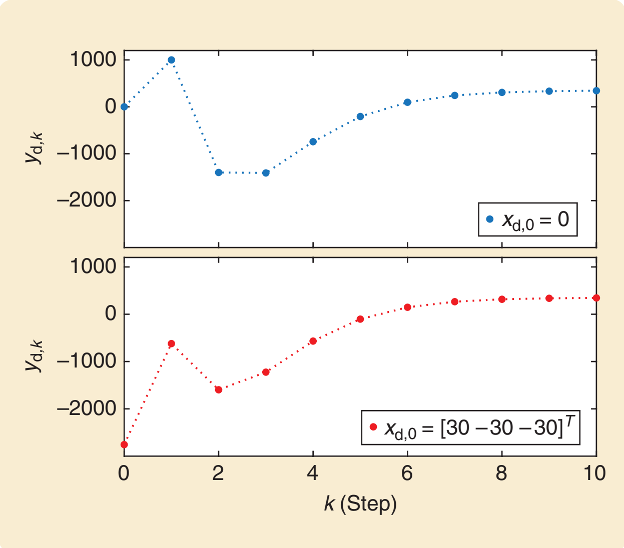

FIGURE 15 Example 14. The response $y_{\mathrm{d},k}$ of (120) for two initial states. With $x_{\mathrm{d},0} = 0$, $\tilde{G}_{\mathrm{d}}(z) = G_{\mathrm{d}}(z)$ has exactly two real zeros greater than one, and $y_{\mathrm{d},k}$ has delayed undershoot, whereas with $x_{\mathrm{d},0} = [30\ -30\ -30]^T$, $\tilde{G}_{\mathrm{d}}(z)$ has no real zeros greater than one, and $y_{\mathrm{d},k}$ does not have delayed undershoot.

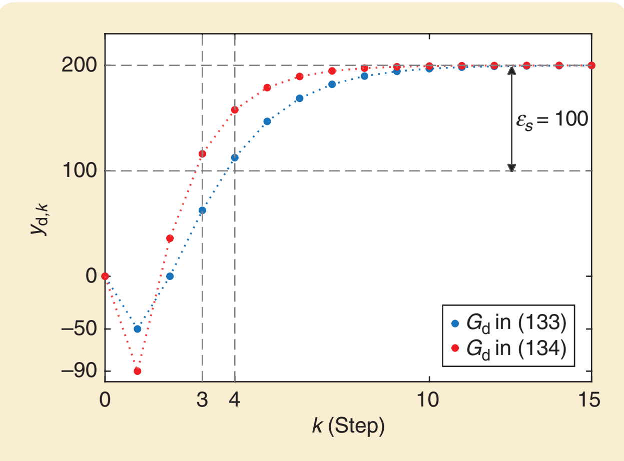

FIGURE 16 Example 15. The unit step responses of (133) and (134), each of which has exactly one real zero at two. Since (133) has slower dynamics than (134), (126) implies that for a given ε_s, the lower bound for δ_{d} in the case of (133) is smaller than the lower bound for δ_{d} in the case of (134).

Proof

Proposition 12 implies that

$$\sum_{k=0}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) = \frac{z_1}{z_1 - 1}\tilde{G}_{\mathrm{d}}(z_1) = 0, \tag{127}$$

where, since $\tilde{G}_{\mathrm{d}}(z)$ is asymptotically stable and $z_1 > 1$, the sum is convergent. Furthermore,

$$\sum_{k=0}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) = \sum_{k=0}^{k_s-1} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) + \sum_{k=k_s}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}),$$

which, combined with (127), implies that

$$\sum_{k=k_s}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) = -\sum_{k=0}^{k_s-1} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}). \tag{128}$$

Since $\tilde{G}_{\mathrm{d}}(1) \neq 0$, Propositions 12 and 13 imply that either $y_{\mathrm{d},\infty} > y_{\mathrm{d},0}$ or $y_{\mathrm{d},\infty} < y_{\mathrm{d},0}$. In the case where $y_{\mathrm{d},\infty} > y_{\mathrm{d},0}$, (128) implies that

$$\begin{aligned} \sum_{k=k_s}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) &\leq \delta_{\mathrm{c}} \sum_{k=0}^{k_s-1} z_1^{-k} \\ &= \frac{\delta_{\mathrm{c}}(z_1 - z_1^{-k_s+1})}{z_1 - 1}. \end{aligned} \tag{129}$$

Moreover, since $y_{\mathrm{d},k} \geq y_{\mathrm{d},\infty} - \varepsilon_s$, for all $k \geq k_s$, it follows that

$$\begin{aligned} \sum_{k=k_s}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) &\geq (y_{\mathrm{d},\infty} - y_{\mathrm{d},0} - \varepsilon_s) \sum_{k=k_s}^{\infty} z_1^{-k} \\ &= \frac{(y_{\mathrm{d},\infty} - y_{\mathrm{d},0} - \varepsilon_s) z_1^{-k_s+1}}{z_1 - 1}. \end{aligned} \tag{130}$$

Combining (129) and (130) yields

$$\frac{y_{\mathrm{d},\infty} - y_{\mathrm{d},0} - \varepsilon_s}{z_1^{k_s} - 1} = \frac{|y_{\mathrm{d},\infty} - y_{\mathrm{d},0}| - \varepsilon_s}{z_1^{k_s} - 1} \leq \delta_{\mathrm{d}}.$$

In the case where $y_{\mathrm{d},\infty} < y_{\mathrm{d},0}$, (128) implies that

$$\begin{aligned} \sum_{k=k_s}^{\infty} z_1^{-k}(y_{\mathrm{d},k} - y_{\mathrm{d},0}) &\geq -\delta_{\mathrm{d}} \sum_{k=0}^{k_s-1} z_1^{-k} \\ &= \frac{\delta_{\mathrm{d}}(z_1^{-k_s+1} - z_1)}{z_1 - 1}. \end{aligned} \tag{131}$$

Moreover, since $y_{\mathrm{d},k} \leq y_{\mathrm{d},\infty} + \varepsilon_s$, for all $k \geq k_s$, it follows that

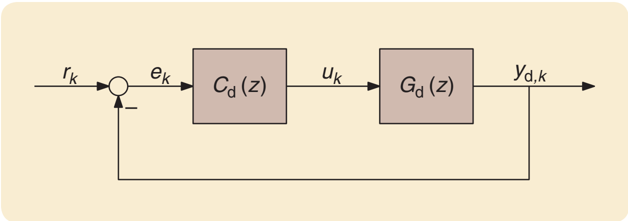

FIGURE 17 A basic servo loop for the discrete-time system (85) and (86).

$$\sum_{k=k_s}^{\infty} z_1^{-k}(y_{d,k}-y_{d,0}) \le (y_{d,\infty}-y_{d,0}+\varepsilon_s)\sum_{k=k_s}^{\infty} z_1^{-k} = \frac{(y_{d,\infty}-y_{d,0}+\varepsilon_s)z_1^{-k_s+1}}{z_1-1}. \quad (132)$$

Combining (131) and (132) yields

$$\frac{y_{d,0}-y_{d,\infty}-\varepsilon_s}{z_1^{k_s}-1} = \frac{|\,y_{d,\infty}-y_{d,0}\,|-\varepsilon_s}{z_1^{k_s}-1} \le \delta_d. \qquad \square$$

In addition to providing a lower bound on δ_d, Theorem 8 implies a relationship between the transient and asymptotic performances of dynamic systems that have delayed undershoot. The following example demonstrate one application of Theorem 8.

Example 15

Consider the transfer functions

$$G_d(z) = \frac{-50(z-2)}{(z-0.5)^2}, \quad (133)$$

$$G_d(z) = \frac{-90(z-2)}{(z-0.5)(z-0.1)}, \quad (134)$$

both of which have exactly one zero at two. Let $\bar{u}=1$ and $x_{d,0}=0$, and note that for both (133) and (134), Proposition 13 implies that $y_{d,\infty}=200$. Moreover, note that (133) has slower dynamics than (134). Let $\varepsilon_s=100$, and define δ_d and k_s by (124) and (125), respectively. Consider the unit step responses of (133) and (134) shown in Figure 16. For (133), $k_s=4$, which, using (126) implies that $\delta_d \ge 6.67$. The lower bound on δ_d is conservative by a factor of about 7.5, as shown in Figure 16, where $\delta_d=50$. Similarly, for (134), $k_s=3$, which [using (126)] implies that $\delta_d \ge 14.29$. The lower bound on δ_d is conservative by a factor of about 6.29, as shown in Figure 16, where $\delta_d=90$. Since (133) has slower dynamics than (134), (126) implies that for a given ε_s, δ_d has a smaller lower bound for (133) than for (134). ◊

To conclude this section, note that the presence or absence of initial and delayed undershoot depends on the initial state, and, thus, initial and delayed undershoot in setpoint command following for discrete-time systems is similar to the continuous-time case. Figure 17 shows a block diagram of the basic servo loop for the discrete-time system (85) and (86), where r_k is the command, $e_k \triangleq r_k - y_{d,k}$ is the error, and $C_d(z)$ is the discrete-time controller.

INITIAL UNDERSHOOT IN SAMPLED-DATA SYSTEMS

The sampled step response of a continuous-time system is the step response of the discrete-time system obtained by discretizing the continuous-time dynamics. This section considers initial undershoot in the sampled-data system to relate the zeros of the sampled-data system to the zeros of the continuous-time system. We use zero-order-hold (ZOH) inputs and instantaneous sampling, which correspond to the Matlab c2d function. Figure 18 shows a block diagram of the basic servo loop for the sampled-data system with sample time T_s, where $y_c(t)$ is the continuous-time output, $y_{d,k}$ is the sampled output, $r(t)$ is the continuous-time command, r_k is the sampled command, $e_k \triangleq r_k - y_{d,k}$ is the error, $C_d(z)$ is the discrete-time controller, u_k is the discrete-time control, and $u(t)$ is the continuous-time control.

Example 16

For the continuous-time, strictly proper transfer function

$$G_c(s) = \frac{-2(s-1)}{s^2+s+2}, \quad (135)$$

with $x_c(0)=0$, Theorem 1 implies that $y_c(t)$ has initial undershoot. Assuming ZOH with sample time $T_s = 1$ s, the discretization of (135) is given by

$$G_d(z) = \frac{-0.2600(z-5.116)}{z^2-0.2977z+0.3679}. \quad (136)$$

Since $G_d(z)$ has exactly one real zero greater than one, Theorem 5 implies that $y_{d,k}$ has initial undershoot. On the other hand, the discretization of (135) with sample time $T_s = 1.5$ s is given by

$$G_d(z) = \frac{0.3724(z+3.304)}{z^2+0.3796z+0.2231}, \quad (137)$$

which has no real zeros greater than one, and, thus, Theorem 5 implies that $y_{d,k}$ does not have initial undershoot.

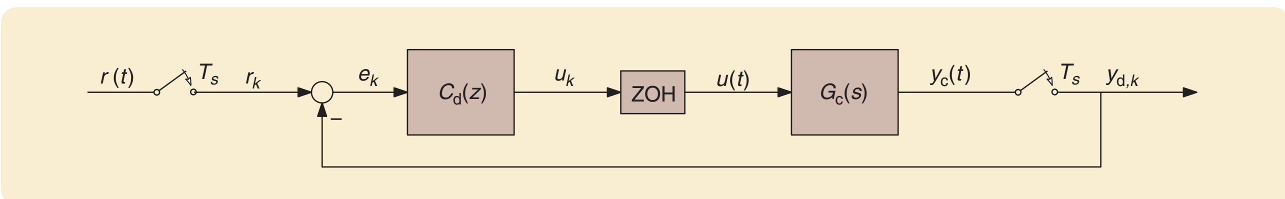

FIGURE 18 A basic servo loop for the sampled-data system with zero-order-hold (ZOH) inputs and instantaneous sampling with sample time T_s.

Figure 19 shows $y_c(t)$ and $y_{d,k}$ for both (136) and (137). This example shows that, for a continuous-time system with a real zero greater than zero, the discretized system may or may not have a real zero greater than one, depending on the sample time T_s. In particular, for sufficiently large T_s, the sampled response may miss significant dynamics, as in the case of aliasing. ◊

Example 17

For the continuous-time, strictly proper transfer function

$$G_c(s) = \frac{2(s-1)(s-2)}{(s+3)(s^2+s+12)}, \tag{138}$$

with $x_c(0) = 0$, Theorem 1 implies that $y_c(t)$ does not have initial undershoot. The discretization of (138) with a ZOH and sample time $T_s = 0.25$ s is given by

$$G_d(z) = \frac{0.1430(z-1.228)(z-2.123)}{(z-0.4724)(z^2-1.156z+0.7788)}. \tag{139}$$

Note that $G_d(z)$ has exactly two real zeros greater than one, and Theorem 5 implies that $y_{d,k}$ does not have initial undershoot. The discretization of (138) with sample time $T_s = 1$ s is given by

$$G_d(z) = \frac{-0.2968(z+0.6255)(z-1.554)}{(z-0.0498)(z^2+1.164z+0.3679)}, \tag{140}$$

which has exactly one real zero greater than one, and, thus, Theorem 5 implies that $y_{d,k}$ has initial undershoot. The discretization of (138) with sample time $T_s = 2$ s is given by

$$G_d(z) = \frac{0.3094(z-0.5385)(z-0.5987)}{(z-0.0025)(z^2-0.6185z+0.1353)}, \tag{141}$$

which has no real zeros greater than one, and, thus, Theorem 5 implies that $y_{d,k}$ does not have initial undershoot. Figure 20 shows $y_c(t)$ and $y_{d,k}$ for (139)–(141). This example shows that, for a lightly damped, continuous-time system with no real zeros greater than zero, the discretized system may have zero, one, or two real zeros greater than one depending on the sample time T_s. ◊

Example 18

For the continuous-time, strictly proper transfer function

$$G_c(s) = \frac{2(s-1)(s-2)}{(s+1)(s^2+s+2)}, \tag{142}$$

with $x_c(0) = 0$, Theorem 1 implies that $y_c(t)$ does not have initial undershoot. The discretization of (142) with a ZOH and sample time $T_s = 0.5$ s is given by

$$G_d(z) = \frac{0.0958(z-6.5)(z-1.564)}{(z-0.6065)(z^2-1.229z+0.6065)}. \tag{143}$$

Since $G_d(z)$ has exactly two real zeros greater than one, Theorem 5 implies that $y_{d,k}$ does not have initial undershoot. The discretization of (142) with sample time $T_s = 1$ s is given by

$$G_d(z) = \frac{-0.5009(z+1.267)(z-2.192)}{(z-0.3679)(z^2-0.2977z+0.3679)}, \tag{144}$$

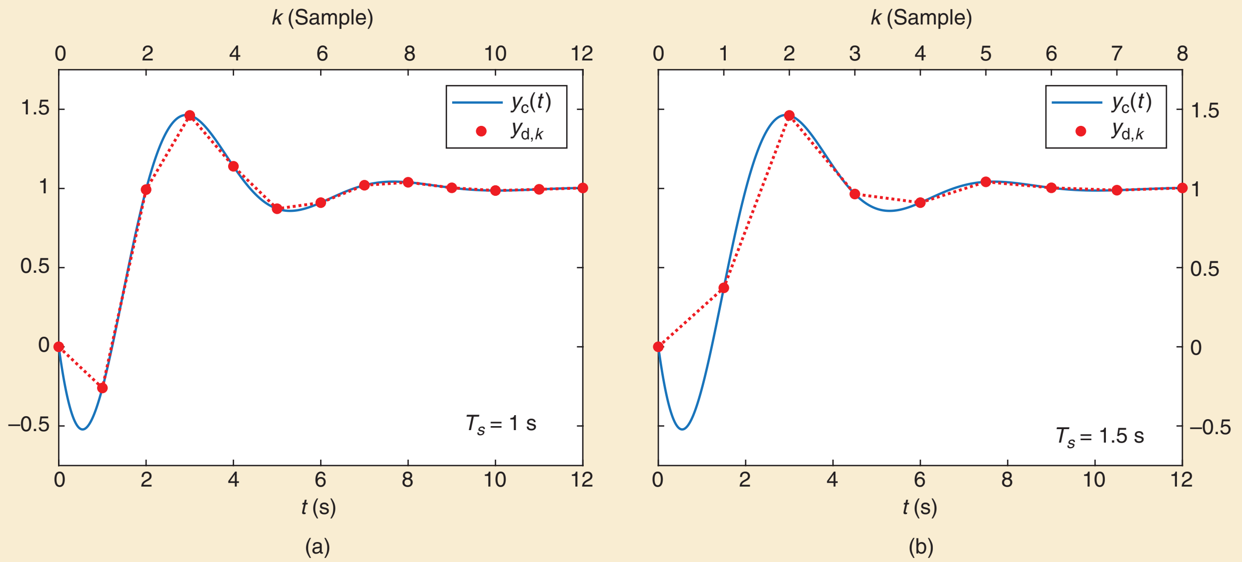

FIGURE 19 Example 16. For zero initial states, (a) shows the unit step response of (135) and (136), and (b) shows the unit step response of (135) and (137). Equation (135) has exactly one real zero greater than zero, and $y_c(t)$ has initial undershoot. Equation (136) has exactly one real zero greater than one, and $y_{d,k}$ has initial undershoot, as shown in (a). However, (137) has no zeros greater than one, and, thus, $y_{d,k}$ does not have initial undershoot, as shown in (b).

which has exactly one real zero greater than one, and Theorem 5 implies that $y_{\mathrm{d},k}$ has initial undershoot. The discretization of (142) with sample time $T_s = 2.5$ s is given by

$$G_{\mathrm{d}}(z) = \frac{0.4486\,(z+0.3108)\,(z+4.143)}{(z-0.0821)\,(z^2+0.5652z+0.0821)}, \qquad (145)$$

which has no real zeros greater than one, and, thus, Theorem 5 implies that $y_{\mathrm{d},k}$ does not have initial undershoot. Figure 21 shows $y_{\mathrm{c}}(t)$ and $y_{\mathrm{d},k}$ for (143)–(145). This example shows that, for a continuous-time system with exactly two real zeros greater than zero, the discretized system may have zero, one, or two real zeros greater than one depending on the sample time T_s. ◊

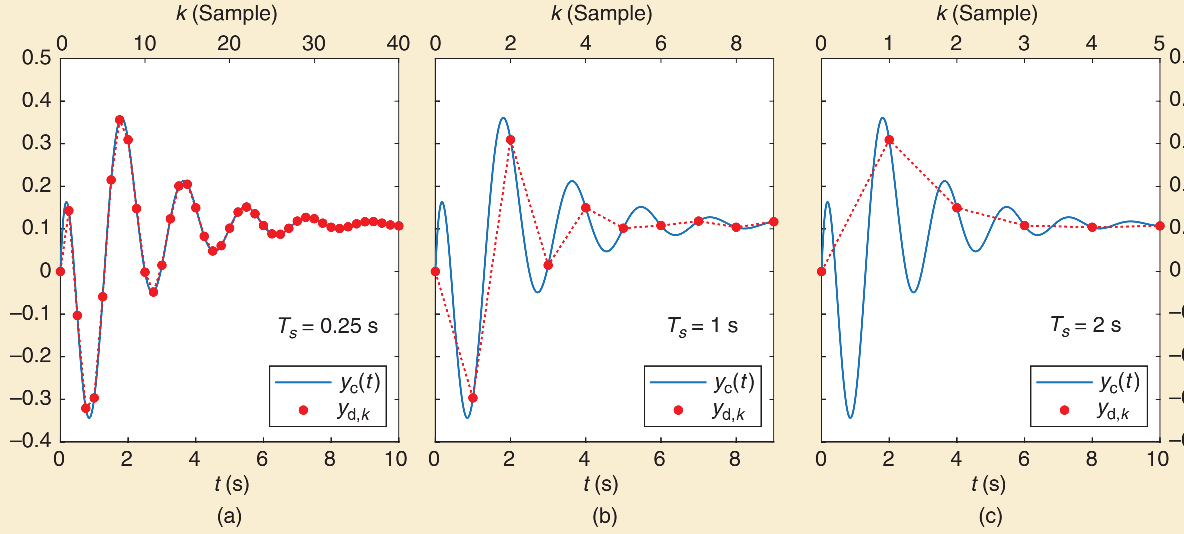

FIGURE 20 Example 17. For zero initial states, (a) shows the unit step response of (138) and (139), (b) shows the unit step response of (138) and (140), and (c) shows the unit step response of (138) and (141). Equation (138) has exactly two real zeros greater than zero, and, thus, $y_{\mathrm{c}}(t)$ does not have initial undershoot. Equation (139) has exactly two real zeros greater than one, and $y_{\mathrm{d},k}$ does not have initial undershoot, as shown in (a). However, (140) has exactly one real zero greater than one, and $y_{\mathrm{d},k}$ has initial undershoot, as shown in (b). Finally, (141) has no real zeros greater than one, and $y_{\mathrm{d},k}$ does not have initial undershoot, as shown in (c).

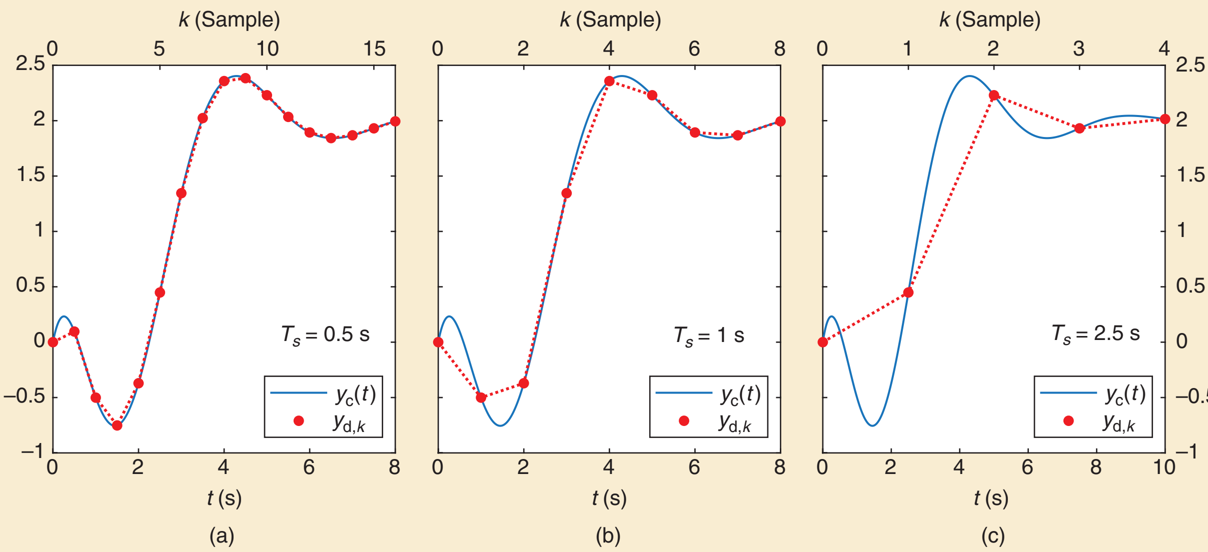

FIGURE 21 Example 18. For zero initial states, (a) shows the unit step response of (142) and (143), (b) shows the unit step response of (142) and (144), and (c) shows the unit step response of (142) and (145). Equation (142) has exactly two real zeros greater than zero, and, thus, $y_{\mathrm{c}}(t)$ does not have initial undershoot. Equation (143) has exactly two real zeros greater than one, and $y_{\mathrm{d},k}$ does not have initial undershoot, as shown in (a). However, (144) has exactly one real zero greater than one, and $y_{\mathrm{d},k}$ has initial undershoot, as shown in (b). Finally, (145) has no real zeros greater than one, and $y_{\mathrm{d},k}$ does not have initial undershoot, as shown in (c).

The following result shows that, under sufficiently fast sampling, the step response of the discretization of a continuous-time system has initial undershoot if and only if the step response of the continuous-time system has initial undershoot.

Theorem 9

Consider the continuous-time system (1) and (2), where $x_c(0) \in \mathbb{R}^n$. Let (85) and (86) be the discretization of (1) and (2) with sample time T_s, where $x_{d,0} = x_c(0)$. Let the transfer functions $G_c(s), \tilde{G}_c(s), G_d(z)$, and $\tilde{G}_d(z)$ be defined by (3), (12), (87), and (97), respectively. Then, there exists $T_0 > 0$ such that, for all $T_s \in (0, T_0]$, the following statements hold:

i) $y_{d,k}$ has initial undershoot if and only if $y_c(t)$ has initial undershoot.

ii) Assume that $\tilde{G}_c(0) \neq 0$. Then, the number of real zeros greater than one of $\tilde{G}_d(z)$ is odd if and only if the number of real zeros greater than zero of $\tilde{G}_c(s)$ is odd.

iii) The number of real zeros greater than one of $G_d(z)$ is equal to the number of real zeros greater than zero of $G_c(s)$.

Proof

To show i) and ii), note that for all $k \geq 0$, $y_{d,k} = y_c(kT_s)$, which implies that there exists $T_0 > 0$ such that for all $T_s \in (0, T_0]$, $y_{d,k}$ has initial undershoot if and only if $y_c(t)$ has initial undershoot. Thus, Theorems 2 and 6 imply that there exists $T_0 > 0$ such that for all $T_s \in (0, T_0]$, $\tilde{G}_d(z)$ has an odd number of real zeros greater than one if and only if $\tilde{G}_c(s)$ has an odd number of real zeros greater than zero.

To show iii), note that since $G_d(z)$ is the discretized version of $G_c(s)$ with sample time T_s, it follows that there exists $T_0 > 0$ such that, for all $T_s \in (0, T_0]$, the zeros of a $G_d(z)$ are approximately equal to $e^{z_i T_s}$, where z_i are the zeros of the $\tilde{G}_c(s)$, and all of the zeros introduced by sampling are negative [17, p. 64]. Therefore, there exists $T_0 > 0$ such that, for all $T_s \in (0, T_0]$, the number of real zeros greater than one of $G_d(z)$ is equal to the number of real zeros greater than one of $G_c(s)$. □

The following result shows that, under sufficiently slow sampling, the response of a sampled-data system does not exhibit delayed undershoot.

Theorem 10

Consider the continuous-time system (1) and (2), where $x_c(0) \in \mathbb{R}^n$. Let (85) and (86) be the discretization of (1) and (2) with sample time T_s, where $x_{d,0} = x_c(0)$. Let the transfer functions $\tilde{G}_c(s)$ and $\tilde{G}_d(z)$ be defined by (12) and (97), respectively, and assume that $\tilde{G}_c(0) \neq 0$. Then, there exists $T_0 > 0$ such that, for all $T_s \geq T_0$, $\tilde{G}_d(z)$ has no real zeros greater than one.

Proof

Since $\tilde{G}_c(0) \neq 0$, Propositions 4 and 5 imply that $y_c(\infty) \neq y_c(0)$. Without loss of generality, we consider the case where $y_c(\infty) > y_c(0)$. It thus follows that there exists $T_0 > 0$ such that, for all $t \geq T_0$, $y_c(t) > y_c(0)$. Thus, for all $t \geq T_0$, $[y_c(t) - y_c(0)][y_c(\infty) - y_c(0)] > 0$. Now, let $T_s \geq T_0$, and note that, for all $k \geq 0$, $y_{d,k} = y_c(kT_s)$. Then, for the discretized system (85) and (86) with sample time T_s, it follows that, for all $k \geq 1$, $(y_{d,k} - y_{d,0})(y_{d,\infty} - y_{d,0}) > 0$, and $y_{d,k}$ does not have delayed undershoot. Therefore, Theorem 7 implies that $\tilde{G}_d(z)$ has no real zeros greater than one. □

CONCLUSIONS AND FUTURE WORK

For the step response with a possibly nonzero initial state, this article presented necessary and sufficient conditions for initial undershoot in continuous- and discrete-time systems. In both cases, it was shown that initial and delayed undershoot depend on the initial condition of the plant. Consequently, for setpoint command following, the internal state at the time at which the setpoint command changes can affect the presence or absence of initial or delayed undershoot; when the state is unknown, this dependence is enigmatic. Undershoot in sampled-data systems was also considered, providing insight into the relationship between the zeros of the underlying continuous-time system and the zeros of the discretized system.

As an extension of this work, the number of sign reversals (that is, zero crossings) of the step response in the case of the zero initial state is of interest [26, p. 184]. Finally, initial undershoot in nonlinear systems with unstable zero dynamics is considered in [10] but relatively unexplored.

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» MEMBER ACTIVITIES *(continued from p. 16)*

of us can take now to increase diversity and build a more inclusive environment. The discussions that followed were very interesting and simulating, and they were a good starting point for more conversations and actions about diversity, inclusion, women empowerment, and the crucial role of technical societies in shaping the future and increasing the visibility and impact of a diverse community.

The webinar was the result of great teamwork. It was widely advertised on the Women in Control webpage and social media accounts, IEEE Control Systems Society (CSS) Twitter account, and ACC conference website. It attracted 48 attendees, including Dr. Thomas Parisini, CSS president; Dr. Anuradha Annaswamy, CSS past president; Dr. Jay Farrell, president of the American Automatic Control Council (AACC); Dr. Maria Domenica Di Benedetto, IEEE CSS vice president of Member Activities; and Dr. George Chiu, ACC2021 general chair.

We are grateful for the speakers who volunteered their time and expertise to engage the attendees in a crucial topic and support this event. We are also grateful for the CSS and AACC support for Women in Control activities. To stay up to date and follow our upcoming events, you are cordially invited to join us on our webpage (http://ieeecss.org/member-activities/women-control), LinkedIn (https://www.linkedin.com/groups/5090839/), and Facebook (https://www.facebook.com/groups/CSS.WiC/) accounts.

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