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On the zero dynamics of linear input–output models

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ABSTRACT

Zeros are of extreme importance in linear systems theory, especially unstable zeros, which degrade achievable performance. Furthermore, the presence of a zero in a state-space model implies the existence of an initial condition and a nonzero input signal such that the output is identically zero; this property is called *output zeroing*. The purpose of this paper is to elucidate the properties of the zero dynamics within the context of input–output models, which, like state-space models, are time-domain models, but, unlike state-space models, have no internal state. In particular, the focus is on the zero dynamics of left polynomial fraction description (LPFD) input–output models whose denominator polynomial is not necessarily monic.

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1. Introduction

Among the various representations of linear systems are state-space models, transfer functions, and input–output models. The first two are widely used, and the relationship between transfer functions and their state-space realisations has been extensively investigated for more than 70 years (Kailath, 1980). Less well known is the class of models that have the form of transfer functions but, like state-space models, operate in the time domain. These are *input–output* models, which are variously known as time series or ARMAX models (Gevers, 1986; Janssen, 1988). Input–output models provide the framework for behaviours, which are essentially time-domain polynomial models in the differential operator (Willems & Polderman, 2013).

In continuous time, the distinction between transfer functions and input–output models resides in the distinction between the Laplace transform variable s and the differentiation operator $\mathbf{p}$. Analogously, in discrete time, this distinction hinges on the distinction between the Z transform variable z and the forward-shift operator $\mathbf{q}$ (Middleton & Goodwin, 1990). One consequence of this distinction is the fact that, unlike transfer function models, an input–output model does not require a separate term to represent the free response (Aljanaideh & Bernstein, 2018).

Regardless of model representation, zeros are of extreme importance in linear systems theory (Desoer & Schulman, 1974; MacFarlane & Karcanias, 1976; Rosenbrock, 1973; Schrader & Sain, 1989; Tokarzewski, 2006), especially unstable zeros, which degrade achievable performance (Havre & Skogestad, 2001; Hoagg & Bernstein, 2007). In particular, if a continuous-time linear system has a zero in the open right half-plane, then the zero dynamics are unstable, and the corresponding input is unbounded. For discrete-time systems, the analogous property holds for zeros contained in the complement of the closed unit disk. For MIMO transfer functions, transmission zeros can be determined from either the Rosenbrock system matrix or the Smith–McMillan form (Kailath, 1980). These zeros coincide with the invariant zeros of a minimal realisation.

A broader framework within which to understand the implications of zeros is the notion of zero dynamics, which is applicable to both linear and nonlinear systems (Berger et al., 2015; Daoutidis & Kravaris, 1991; Isidori, 2013). Zero dynamics are the 'dynamics' of the input assuming that the output is identically zero. Of course, for a state-space model, if the initial condition and input are both zero, then the output is zero. However, the zero dynamics have the interesting property that there exist a nonzero input and a nonzero initial condition such that the output is identically zero; this phenomenon is called *output zeroing* (Callier & Desoer, 2012; Desoer & Schulman, 1974; Karampetakis, 1998; Karcanias & Kouvaritakis, 1979; Tokarzewski, 2006). Along with basic results on output zeroing, Karcanias and Kouvaritakis (1979) explores the zero structure using zero pencils and describes the geometric properties of the zero structure. Zeros of discrete-time systems are discussed in detail in Tokarzewski (2006). The above-mentioned works on output zeroing consider state-space models; input–output models are not considered.

The contribution of the present article is an elucidation of the properties of the zero dynamics of input–output models as well as an exploration of output zeroing in input–output models. To this end, the paper has two main objectives. The first objective is to characterise the transmission zeros using left polynomial fraction description (LPFD) input–output models, define the zero dynamics of these models, and describe the solutions of the zero dynamics of these models that correspond to the transmission zeros of the system.

The second objective is to establish conditions under which output zeroing is achieved in input–output models and to connect output zeroing in input–output models to output zeroing in state-space models. First, output zeroing using a *monic* LPFD

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is connected to output zeroing using the observable canonical form realisation corresponding to the given monic LPFD. Next, the equivalence between output zeroing using a *coprime* (and not necessarily monic) LPFD and output zeroing using an arbitrary minimal realisation is established. This result is based on the construction of a minimal state-space realisation based on a coprime LPFD as given in Polak (1966). The paper concludes with an example that illustrates the equivalence between output zeroing in input–output models and output zeroing in state-space models.

2. Preliminaries

Let $\mathbb{R}[z]^{p\times m}$ denote the set of $p \times m$ matrices each of whose entries is a polynomial with real coefficients, let $\mathbb{R}(z)^{p\times m}$ denote the set of $p \times m$ matrices each of whose entries is a rational function with real coefficients, and let $\mathbb{R}(z)^{p\times m}_{\rm prop}$ denote the proper transfer functions in $\mathbb{R}(z)^{p\times m}$. In this and all subsequent sections, let $G \in \mathbb{R}(z)^{p\times m}_{\rm prop}$. The *rank* of G is the maximum value of $\operatorname{rank} G(z)$ taken over the set of complex numbers z such that, for all $i = 1, \ldots, p$ and $j = 1, \ldots, m$, z is not a pole of the (i, j) entry of G. The McMillan degree of G is denoted by McDeg G. Throughout this paper, $0^0 \triangleq 1$.

The following result given by Theorem 6.7.5 in Bernstein (2018, p. 514) presents the *Smith–McMillan form* S of G.

Theorem 2.1: *There exist unimodular matrices $S_1 \in \mathbb{R}[z]^{p\times p}$ and $S_2 \in \mathbb{R}[z]^{m\times m}$ and unique monic polynomials $p_1, \ldots, p_\rho$, $q_1, \ldots, q_\rho \in \mathbb{R}[z]$ such that, for all $i \in \{1, \ldots, \rho\}$, p_i and q_i are coprime, for all $i \in \{1, \ldots, \rho - 1\}$, p_i divides p_{i+1} and q_{i+1} divides q_i, and $G = S_1 S S_2$, where $\rho \triangleq \operatorname{rank} G$ and*

$$S \triangleq \begin{bmatrix} p_1/q_1 & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & \\ & & p_\rho/q_\rho & \\ 0_{(p-\rho)\times\rho} & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \quad (1)$$

The roots of the polynomial $q_1 q_2 \cdots q_\rho$ are the *poles* of G, and the roots of the polynomial $p_1 p_2 \cdots p_\rho$ are the *transmission zeros* of G.

In the notation of Theorem 2.1, define

$$D_{\rm S} \triangleq \begin{bmatrix} q_1 & & & 0 \\ & \ddots & & \\ & & q_\rho & \\ 0 & & & I_{p-\rho} \end{bmatrix} S_1^{-1}, \quad (2)$$

$$N_{\rm S} \triangleq \begin{bmatrix} p_1 & & & 0 \\ & \ddots & & \\ & & p_\rho & \\ 0 & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix} S_2. \quad (3)$$

Definition 2.2: Let $P \in \mathbb{R}[z]^{p\times m}$ and $R \in \mathbb{R}[z]^{p\times p}$. Then R *left divides* P if there exists $\hat{P} \in \mathbb{R}[z]^{p\times m}$ such that $P = R\hat{P}$.

Definition 2.3: Let $U \in \mathbb{R}[z]^{n\times n}$. Then U is *unimodular* if $\det U$ is a nonzero real number.

Definition 2.4: Let $P \in \mathbb{R}[z]^{p\times n}$ and $Q \in \mathbb{R}[z]^{p\times m}$. Then P and Q are *coprime* if every $R \in \mathbb{R}[z]^{p\times p}$ that left divides both P and Q is unimodular.

Definition 2.5: Let $P \in \mathbb{R}[z]^{p\times m}$. Then $\deg P$ is the maximum degree of the entries of P. Furthermore, P is *monic* if $p = m$ and $P(z) = z^{\deg P} I_m + P_0(z)$, where $P_0 \in \mathbb{R}[z]^{m\times m}$ and $\deg P_0 < \deg P$.

Definition 2.6: Let $D \in \mathbb{R}[z]^{p\times p}$, and $N \in \mathbb{R}[z]^{p\times m}$, assume that D is nonsingular, and assume that $G = D^{-1}N$. Then (D, N) is a *left polynomial fraction description (LPFD)* of G. Furthermore, if D and N are coprime, then (D, N) is a *coprime left polynomial fraction description (CLPFD)* of G. In addition, if D is monic, then (D, N) is a *monic left polynomial fraction description (MLPFD)* of G. Finally, if D and N are coprime and D is monic, then (D, N) is a *monic coprime left polynomial fraction description (MCLPFD)* of G.

Note that the terms 'matrix fraction description' and 'polynomial matrix fraction description' are used as alternatives to the term 'polynomial fraction description' in the literature.

Proposition 2.7: *$(D_{\rm S}, N_{\rm S})$ is a CLPFD of G.*

Proof: Note that, for all $z \in \mathbb{C}$, $\operatorname{rank}[D_{\rm S}(z)\ \ N_{\rm S}(z)] = p$, and hence, it follows from Theorem 16.16 in Rugh (1996, p. 300) that $D_{\rm S}$ and $N_{\rm S}$ are coprime. ∎

The following result is given by Theorem 16.17 in Rugh (1996, p. 301).

Proposition 2.8: *Let (D, N) be a CLPFD of G and $(\hat{D}, \hat{N})$ be an LPFD of G. Then $(\hat{D}, \hat{N})$ is a CLPFD of G if and only if there exists a unimodular matrix $U \in \mathbb{R}[z]^{p\times p}$ such that $\hat{D} = UD$ and $\hat{N} = UN$.*

Corollary 2.9: *Let (D, N) be an LPFD of G. Then (D, N) is a CLPFD of G if and only if there exists a unimodular matrix $U \in \mathbb{R}[z]^{p\times p}$ such that $D = UD_{\rm S}$ and $N = UN_{\rm S}$.*

Definition 2.10: Let $P \in \mathbb{R}[z]^{p\times n}$, $Q \in \mathbb{R}[z]^{p\times m}$ and $R \in \mathbb{R}[z]^{p\times p}$. Then R is a *greatest common left divisor* of P and Q if there exists $\hat{P} \in \mathbb{R}[z]^{p\times n}$, $\hat{Q} \in \mathbb{R}[z]^{p\times m}$ such that $P = R\hat{P}$, $Q = R\hat{Q}$ and $\hat{P}$ and $\hat{Q}$ are coprime.

Note that it follows from MacDuffee (2012, p. 35) that, for all $P \in \mathbb{R}[z]^{p\times n}$ and $Q \in \mathbb{R}[z]^{p\times m}$, there exists a greatest common left divisor of P and Q.

Lemma 2.11: *Let (D, N) be a CLPFD of G, and let $(\hat{D}, \hat{N})$ be an LPFD of G. Then there exists a nonsingular $L \in \mathbb{R}[z]^{p\times p}$ such that $\hat{D} = LD$ and $\hat{N} = LN$.*

Proof: Let $R \in \mathbb{R}[z]^{p\times p}$ be a greatest common left divisor of $\hat{D}$ and $\hat{N}$. Then, there exist $\overline{D} \in \mathbb{R}[z]^{p\times p}$ and $\overline{N} \in \mathbb{R}[z]^{p\times m}$ such that $\hat{D} = R\overline{D}$, $\hat{N} = R\overline{N}$, and $\overline{D}$ and $\overline{N}$ are coprime. Next, it follows from Proposition 2.8 that there exists a unimodular matrix $U \in \mathbb{R}[z]^{p\times p}$ such that $\overline{D} = UD$ and $\overline{N} = UN$. Hence, $\hat{D} = LD$

and $\hat{N} = LN$, where $L \triangleq RU$. Since $\hat{D}$ is nonsingular, it follows that R is nonsingular, and thus, L is nonsingular. ■

Proposition 2.12: *Let (D, N) be an LPFD of G. Then* $\deg\det D = \operatorname{McDeg} G$ *if and only if (D, N) is a CLPFD of G.*

Proof: To prove sufficiency, note that Corollary 2.9 implies that there exists a unimodular matrix $U \in \mathbb{R}[z]^{p\times p}$ such that $D = UD_{\rm S}$. Hence, $\deg\det D = \deg\det U + \deg\det D_{\rm S} = \deg\det D_{\rm S} = \operatorname{McDeg} G$. To prove necessity, note that it follows from Lemma 2.11 and Proposition 2.7 that there exists a nonsingular $L \in \mathbb{R}[z]^{p\times p}$ such that $D = LD_{\rm S}$ and $N = LN_{\rm S}$.

Hence $\deg\det L = \deg\det D - \deg\det D_{\rm S} = \operatorname{McDeg} G - \operatorname{McDeg} G = 0$. Thus, L is unimodular and therefore Corollary 2.9 implies that (D, N) is a CLPFD of G. ■

3. Zero dynamics of input–output models

This section discusses various aspects of the zero dynamics of input–output models. In particular, Proposition 3.1 characterises transmission zeros of G using an LPFD of G and a CLPFD of G. Next, Proposition 3.2 gives an expression for counting the number of transmission zeros of G using a CLPFD of G. The zero dynamics of G are defined in Definition 3.3, necessary and sufficient conditions for the existence of nonzero solutions to the zero dynamics are given in Proposition 3.4, and solutions of the zero dynamics are characterised by Proposition 3.5. Next, Theorem 3.6 relates nonzero solutions of the zero dynamics to the transmission zeros of G.

Proposition 3.1: *Let (D, N) be an LPFD of G, and let z_0 be a transmission zero of G. Then* $\operatorname{rank} N(z_0) < \operatorname{rank} N$. *Now assume that (D, N) is a CLPFD of G. Then z_0 is a transmission zero of G if and only if* $\operatorname{rank} N(z_0) < \operatorname{rank} N$.

Proof: To prove the first statement, note that Proposition 2.7 and Lemma 2.11 imply that there exists a nonsingular $L \in \mathbb{R}[z]^{p\times p}$ such that $D = LD_{\rm S}$ and $N = LN_{\rm S}$, where $D_{\rm S}$ and $N_{\rm S}$ are defined in (2) and (3). Since z_0 is a transmission zero of G, it follows from Theorem 2.1 that $\operatorname{rank} N_{\rm S}(z_0) < \operatorname{rank} N_{\rm S}$. Hence, $\operatorname{rank} N(z_0) \le \operatorname{rank} N_{\rm S}(z_0) < \operatorname{rank} N_{\rm S} = \operatorname{rank} N$. To prove sufficiency in the second statement, note that Corollary 2.9 implies that, for all $z \in \mathbb{C}$, $\operatorname{rank} N(z) = \operatorname{rank} N_{\rm S}(z)$. Hence, $\operatorname{rank} N_{\rm S}(z_0) = \operatorname{rank} N(z_0) < \operatorname{rank} N = \operatorname{rank} N_{\rm S}$, and thus, it follows from Theorem 2.1 that z_0 is a transmission zero of G. ■

Proposition 3.2: *Let (D, N) be a CLPFD of G, let $(\hat{D}, \hat{N})$ be a CLPFD of $G^{\rm T}$, and let ζ be the number of transmission zeros of G counting multiplicity.*

(i) *If* $\operatorname{rank} G = p$, *then* $\zeta = \frac{1}{2}\deg\det NN^{\rm T}$.
(ii) *If* $\operatorname{rank} G = m$, *then* $\zeta = \frac{1}{2}\deg\det \hat{N}\hat{N}^{\rm T}$.

Proof: To prove (i), note that it follows from Corollary 2.9 that $NN^{\rm T} = UN_{\rm S}N_{\rm S}^{\rm T}U^{\rm T}$, where $U \in \mathbb{R}[z]^{p\times p}$ is a unimodular matrix. Since $\operatorname{rank} G = p$, it follows that $N_{\rm S}N_{\rm S}^{\rm T}$ is nonsingular, and thus, $NN^{\rm T}$ is nonsingular. Thus, $\deg\det NN^{\rm T} = \deg\det N_{\rm S} N_{\rm S}^{\rm T} = 2\zeta$. To prove (ii), note that the number of transmission zeros of $G^{\rm T}$ is equal to the number of transmission zeros of G. Since $\operatorname{rank} G^{\rm T} = \operatorname{rank} G = m$, applying (i) to $G^{\rm T}$ yields (ii). ■

The following definition defines the zero dynamics of an LPFD of G and the zero dynamics of G.

Definition 3.3: Let (D, N) be an LPFD of G, and for all $k \ge 0$, let $u_k \in \mathbb{C}^m$ satisfy

$$N(\mathbf{q})u_k = 0. \tag{4}$$

Then, (4) is the *zero dynamics of* (D, N). If, in addition, (D, N) is a CLPFD of G, then (4) is the *zero dynamics of* G.

The following result gives necessary and sufficient conditions for the existence of nonzero solutions of (4).

Proposition 3.4: *Let $N \in \mathbb{R}[z]^{p\times m}$. Then (4) has a nonzero solution if and only if there exists $z_0 \in \mathbb{C}$ such that* $\operatorname{rank} N(z_0) < m$.

Proof: To prove sufficiency, let $N(\mathbf{q}) = \mathbf{q}^\ell B_0 + \mathbf{q}^{\ell-1}B_1 + \cdots + B_\ell$. Then

$$\begin{aligned} N(\mathbf{q})z_0^k &= \left(\mathbf{q}^\ell B_0 + \mathbf{q}^{\ell-1}B_1 + \cdots + B_\ell\right) z_0^k \\ &= z_0^{k+\ell}B_0 + z_0^{k+\ell-1}B_1 + \cdots + z_0^k B_\ell \\ &= \left(z_0^\ell B_0 + z_0^{\ell-1}B_1 + \cdots + B_\ell\right) z_0^k \\ &= N(z_0)z_0^k. \end{aligned}$$

Note that there exists $\overline{u} \ne 0$ such that $N(z_0)\overline{u} = 0$. For all $k \ge 0$, define $u_k \triangleq z_0^k\overline{u}$. Hence, for all $k \ge 0$, $N(\mathbf{q})u_k = N(\mathbf{q})z_0^k\overline{u} = N(z_0)z_0^k\overline{u} = z_0^k N(z_0)\overline{u} = 0$. Since $\overline{u} \ne 0$, it follows that u is a nonzero solution of (4).

To prove necessity, note that, in the case where $\operatorname{rank} N < m$, it follows that, for all $z_0 \in \mathbb{C}$, $\operatorname{rank} N(z_0) < m$. In the case where $\operatorname{rank} N = m$, there exists a unimodular matrix $U \in \mathbb{R}[\mathbf{q}]^{p\times p}$ such that $\overline{N} \triangleq UN = \begin{bmatrix} N_0 \\ 0_{(p-m)\times m}\end{bmatrix}$, where $N_0 \in \mathbb{R}[\mathbf{q}]^{m\times m}$ is nonsingular. Then (4) implies that $\overline{N}(\mathbf{q})u_k = U(\mathbf{q})N(\mathbf{q})u_k = 0$, and thus $N_0(\mathbf{q})u_k = 0$. Now, suppose that N_0 is unimodular. Then $N_0^{-1}(\mathbf{q})$ is a polynomial matrix, and hence (4) is equivalent to $N_0^{-1}(\mathbf{q})N_0(\mathbf{q})u_k = 0$, and thus, for all $k \ge 0$, $u_k = 0$, which is a contradiction. It thus follows that N_0 is not unimodular, that is, $\det N_0$ is a nonconstant polynomial in $\mathbf{q}$. ■

The following result characterises the possibly complex solutions of (4).

Proposition 3.5: *Let $z_0 \in \mathbb{C}$, and let $\overline{u} \in \mathbb{C}^m$. Then, for all $k \ge 0$, $u_k \triangleq z_0^k\overline{u}$ satisfies (4) if and only if $N(z_0)\overline{u} = 0$.*

Proof: Let $N(\mathbf{q}) = \mathbf{q}^{\ell} B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_{\ell}$. Then

$$\begin{aligned} N(\mathbf{q}) z_0^k &= \left(\mathbf{q}^{\ell} B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_{\ell}\right) z_0^k \\ &= z_0^{k+\ell} B_0 + z_0^{k+\ell-1} B_1 + \cdots + z_0^k B_{\ell} \\ &= \left(z_0^{\ell} B_0 + z_0^{\ell-1} B_1 + \cdots + B_{\ell}\right) z_0^k \\ &= N(z_0) z_0^k. \end{aligned}$$

To prove sufficiency, note that, for all $k \geq 0$, $N(\mathbf{q}) u_k = N(\mathbf{q}) z_0^k \overline{u} = N(z_0) z_0^k \overline{u} = z_0^k N(z_0) \overline{u} = 0$. To prove necessity, note that, for all $k \geq 0, 0 = N(\mathbf{q}) u_k = z_0^k N(z_0) \overline{u}$. Letting $k = 0$ yields $N(z_0)\overline{u} = 0$. ■

Proposition 3.4 and Proposition 3.5 discuss solutions of (4) in relation to an arbitrary complex number z_0. Since the focus of this paper is on transmission zeros, we now give a result on the relationship between the solutions of (4) and a transmission zero z_0 of G.

Theorem 3.6: *Let (D, N) be an LPFD of G. The following statements hold:*

(i) *If $\operatorname{rank} N < m$, then, for all $z_0 \in \mathbb{C}$, there exists a nonzero $\overline{u} \in \mathcal{N}(N(z_0))$, and for all $k \geq 0$, $u_k \triangleq z_0^k \overline{u}$ is a nonzero solution of (4).*
(ii) *If $\operatorname{rank} N = m$, and $z_0 \in \mathbb{C}$ is a transmission zero of G, then there exists a nonzero $\overline{u} \in \mathcal{N}(N(z_0))$, and for all $k \geq 0$, $u_k \triangleq z_0^k \overline{u}$ is a nonzero solution of (4).*
(iii) *If $\operatorname{rank} N = m$, and (D, N) is a CLPFD of G, then the following statements are equivalent.*
   (a) *$z_0 \in \mathbb{C}$ is a transmission zero of G.*
   (b) *There exists a nonzero $\overline{u} \in \mathcal{N}(N(z_0))$.*
   (c) *There exists a nonzero solution of (4).*

   If these conditions hold, then, for all $k \geq 0$, $u_k \triangleq z_0^k \overline{u}$ is a nonzero solution of (4).

Proof: (i) follows from Proposition 3.5, and (ii) follows from Propositions 3.1 and 3.5. (a) ⇒ (b) in (iii) follows from Proposition 3.1, and (b) ⇒ (c) in (iii) follows from Proposition 3.5. To prove (c) ⇒ (a) in (iii), note that Proposition 3.4 implies that there exists $z_0 \in \mathbb{C}$ such that $\operatorname{rank} N(z_0) < m = \operatorname{rank} N$. Thus, Proposition 3.1 implies that z_0 is a transmission zero of G. ■

4. Equivalence of output zeroing in input–output and state space models

If G has a transmission zero, then it follows from Tokarzewski (2006, p. 25) that there exist an initial condition and a nonzero input such that the response of a minimal state-space realisation of G is identically zero. This is called *output zeroing* in state-space models. Proposition 4.2 and Corollary 4.3 deal with output zeroing in state-space models, and Theorem 4.4 relates output zeroing in state-space models to the transmission zeros of G. In contrast, Theorem 4.6 and Corollary 4.7 discuss output zeroing in input–output models. Next, Theorem 4.8 relates output zeroing in input–output models to the transmission zeros of G, where it is shown that, if G has a transmission zero, then there exists a nonzero input such that the response of a time-domain input–output representation of G is identically zero. Furthermore, this section connects output zeroing in input–output models to output zeroing in state-space models.

In particular, Theorems 4.10 and 4.14 establish the equivalence between output zeroing in input–output models and output zeroing in state-space models. Finally, an example is given to illustrate this equivalence.

Given a realisation (A, B, C, E) of G, define

$$\mathcal{Z}(z) \triangleq \begin{bmatrix} zI - A & -B \\ C & E \end{bmatrix}. \tag{5}$$

The following result is an immediate consequence of the definition of invariant zeros.

Proposition 4.1: *Let (A, B, C, E) be a realisation of G, where $A \in \mathbb{R}^{n \times n}$. Then, the following statements hold:*

(i) *Assume that $\operatorname{rank} \mathcal{Z} < n + m$. Then, for all $z_0 \in \mathbb{C}$, there exists nonzero $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.*
(ii) *Assume that $\operatorname{rank} \mathcal{Z} = n + m$. Then, $z_0 \in \mathbb{C}$ is an invariant zero of (A, B, C, E) if and only if there exists nonzero $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.*
(iii) *Assume that $\operatorname{rank} \mathcal{Z} = n + m$ and (A, B, C, E) is minimal. Then, $z_0 \in \mathbb{C}$ is a transmission zero of G if and only if there exists nonzero $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.*

The following result on output zeroing in state-space models is given by Lemmas 2.7 and 2.9 in Tokarzewski (2006, pp. 25, 31).

Proposition 4.2: *Let (A, B, C, E) be a realisation of G, where $A \in \mathbb{R}^{n \times n}$, and let $z_0 \in \mathbb{C}$, $\overline{x} \in \mathbb{C}^n$, and $\overline{u} \in \mathbb{C}^m$. Furthermore, define $x_0 \triangleq \overline{x}$, and for all $k \geq 0$, define $u_k \triangleq z_0^k \overline{u}$ and consider*

$$x_{k+1} = A x_k + B u_k, \tag{6}$$

$$y_k = C x_k + E u_k. \tag{7}$$

Then, the following statements hold:

(i) *If $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$, then, for all $k \geq 0$, $y_k = 0$.*
(ii) *If (A, C) is observable and, for all $k \geq 0$, $y_k = 0$, then $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.*

In Proposition 4.2, the signal u and initial state x_0 are not necessarily real. In practice, however, it is desirable to consider real input signals and real states. For this case, the following result is a consequence of statement (i) of Proposition 4.2.

Corollary 4.3: *Let (A, B, C, E) be a realisation of G, where $A \in \mathbb{R}^{n \times n}$, and let $z_0 \in \mathbb{C}$ and $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.*

Define $x_0 \triangleq \operatorname{Re}(\overline{x})$, and for all $k \geq 0$, define $u_k \triangleq \operatorname{Re}(z_0^k \overline{u})$ and consider (6) and (7). Then, for all $k \geq 0$, $y_k = 0$.

The following result relates output zeroing in state-space models to transmission zeros of G.

Theorem 4.4: *Let (A, B, C, E) be a realisation of G, where $A \in \mathbb{R}^{n\times n}$, and let z_0 be a transmission zero of G. Then there exists $\left[\begin{smallmatrix}\bar{x}\\ \bar{u}\end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$, where $\bar{x} \neq 0$ and $\bar{u} \neq 0$. Furthermore, there exist $x_0 \neq 0$ and $u \neq 0$ such that $y \equiv 0$, where x_0, u, and y satisfy* (6) and (7), *and where x_0 and u are real.*

Proof: It follows from the Kalman decomposition (see Proposition 16.9.12 in Bernstein, 2018, p. 1273 or Chapters 2 and 6 in Kailath, 1980) that there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that

$$A_{\rm d} \triangleq SA_{\rm ocf}S^{-1} = \begin{bmatrix} A_1 & 0 & A_{13} & 0\\ A_{21} & A_2 & A_{23} & A_{24}\\ 0 & 0 & A_3 & 0\\ 0 & 0 & A_{43} & A_4\end{bmatrix},$$

$$B_{\rm d} \triangleq SB_{\rm ocf} = \begin{bmatrix} B_1\\ B_2\\ 0\\ 0\end{bmatrix},$$

$$C_{\rm d} \triangleq C_{\rm ocf}S^{-1} = \begin{bmatrix} C_1 & 0 & C_3 & 0\end{bmatrix},$$

where, for all $i = 1, \ldots, 4$, $A_i \in \mathbb{R}^{n_i\times n_i}$, and (A_1, B_1, C_1, E) is a minimal realisation of G. Define $\mathcal{Z}_1(z) \triangleq \left[\begin{smallmatrix} zI-A_1 & -B_1\\ C_1 & E\end{smallmatrix}\right]$. Since z_0 is a transmission zero of G, it follows that $\operatorname{rank}\mathcal{Z}_1(z_0) < \operatorname{rank}\mathcal{Z}_1$. Let $z_1 \in \mathbb{C}$ be such that $\operatorname{rank}\mathcal{Z}_1(z_1) = \operatorname{rank}\mathcal{Z}_1$. Hence $\operatorname{rank}\mathcal{Z}_1(z_0) < \operatorname{rank}\mathcal{Z}_1(z_1)$, and thus Fact 3.14.15 in Bernstein (2018, p. 322) implies that

$$\begin{aligned}&\operatorname{rank}\begin{bmatrix} z_0I-A_1\\ C_1\end{bmatrix} + \operatorname{rank}\begin{bmatrix} B_1\\ E\end{bmatrix}\\ &\quad - \dim\left(\mathcal{R}\left(\begin{bmatrix} z_0I-A_1\\ C_1\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix} B_1\\ E\end{bmatrix}\right)\right)\\ &< \operatorname{rank}\begin{bmatrix} z_1I-A_1\\ C_1\end{bmatrix} + \operatorname{rank}\begin{bmatrix} B_1\\ E\end{bmatrix}\\ &\quad - \dim\left(\mathcal{R}\left(\begin{bmatrix} z_1I-A_1\\ C_1\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix} B_1\\ E\end{bmatrix}\right)\right).\end{aligned}$$

Since (A_1, C_1) is observable, it follows that $\operatorname{rank}\left[\begin{smallmatrix} z_0I-A_1\\ C_1\end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix} z_1I-A_1\\ C_1\end{smallmatrix}\right] = n_1$. Hence,

$$\begin{aligned}&\dim\left(\mathcal{R}\left(\begin{bmatrix} z_0I-A_1\\ C_1\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix} B_1\\ E\end{bmatrix}\right)\right)\\ &\quad > \dim\left(\mathcal{R}\left(\begin{bmatrix} z_1I-A_1\\ C_1\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix} B_1\\ E\end{bmatrix}\right)\right) \geq 0.\end{aligned}$$

Thus, there exists $\left[\begin{smallmatrix}\bar{x}_1\\ \bar{u}_1\end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}_1(z_0))$, where $\bar{x}_1 \neq 0$ and $\bar{u}_1 \neq 0$. Define $\bar{x} \triangleq \left[\begin{smallmatrix}\bar{x}_1\\ 0\end{smallmatrix}\right] \in \mathbb{C}^n$ and $\bar{u} \triangleq \bar{u}_1$. Then, $\bar{x} \neq 0$, $\bar{u} \neq 0$, and $\left[\begin{smallmatrix}\bar{x}\\ \bar{u}\end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$.

Without loss of generality, let $\left[\begin{smallmatrix}\bar{x}\\ \bar{u}\end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$, where $\mathrm{Re}(\bar{x}) \neq 0$ and $\bar{u} \neq 0$. Define $x_0 \triangleq \mathrm{Re}(\bar{x})$ and, for all $k \geq 0$, $u_k \triangleq \mathrm{Re}(z_0^k\bar{u})$. In the case where $\mathrm{Re}(\bar{u}) \neq 0$, note that $u_0 = \mathrm{Re}(\bar{u}) \neq 0$, and thus $u \neq 0$. In the case where $\mathrm{Re}(\bar{u}) = 0$, suppose that $\mathrm{Im}(z_0) = 0$. Then, it follows from $\mathrm{Re}(\mathcal{Z}(z_0)\left[\begin{smallmatrix}\bar{x}\\ \bar{u}\end{smallmatrix}\right]) = 0$ that $\left[\begin{smallmatrix} z_0I-A\\ C\end{smallmatrix}\right]\mathrm{Re}(\bar{x}) = 0$, and thus $\mathrm{Re}(\bar{x}) = 0$, which is a contradiction. Hence, $\mathrm{Im}(z_0) \neq 0$, which implies that $u_1 = \mathrm{Re}(z_0\bar{u}) = \mathrm{Im}(z_0)\mathrm{Im}(\bar{u}) \neq 0$, and thus $u \neq 0$. Finally, Corollary 4.3 implies that, for all $k \geq 0$, $y_k = 0$. ■

Theorem 4.6 concerns output zeroing in input–output models. The proof of this result takes advantage of the following lemma.

Lemma 4.5: *Let $D_0, D_1, \ldots, D_\ell \in \mathbb{R}^{p\times p}$, assume that $D_\ell \neq 0$, and, for all $k \geq 0$, consider the difference equation*

$$D_\ell y_{k+\ell} + \cdots + D_1 y_{k+1} + D_0 y_k = 0, \tag{8}$$

with the initial condition $y_0 = y_1 = \cdots = y_{\ell-1} = 0$.

If $\det(z^\ell D_\ell + \cdots + zD_1 + D_0) \neq 0$, then, for all $k \geq \ell$, $y_k = 0$.

Proof: For all $i < 0$, define $D_i \triangleq 0$, and for all $k \geq 0$, define

$$\mathcal{T}_k \triangleq \begin{bmatrix} D_\ell & 0 & \cdots & 0\\ D_{\ell-1} & D_\ell & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ D_{\ell-k} & D_{\ell-k+1} & \cdots & D_\ell\end{bmatrix} = \begin{bmatrix} Q_k & P_k\end{bmatrix},$$

where

$$Q_k \triangleq \begin{bmatrix} D_\ell\\ D_{\ell-1}\\ \vdots\\ D_{\ell-k}\end{bmatrix}, \quad P_k \triangleq \begin{bmatrix} 0\\ \mathcal{T}_{k-1}\end{bmatrix},$$

and $\mathcal{T}_{-1}$ is the empty matrix.

Next, define $D(z) \triangleq z^\ell D_\ell + \cdots + zD_1 + D_0$ and $G(z) \triangleq D(1/z)$. Note that G is a proper finite-impulse-response (FIR) transfer function, and $D_\ell, \ldots, D_0$ are the Markov parameters of G. Since D is nonsingular, it follows that G is invertible, and hence, it follows from Proposition 2 in Ansari and Bernstein (2019) that there exists $d \geq 0$ such that $\operatorname{rank}\mathcal{T}_d - \operatorname{rank}\mathcal{T}_{d-1} = p$. Next, it follows from Fact 3.14.15 in Bernstein (2018, p. 322) that $p = \operatorname{rank}\mathcal{T}_d - \operatorname{rank}\mathcal{T}_{d-1} = \operatorname{rank}Q_d - \dim(\mathcal{R}(Q_d)\cap\mathcal{R}(P_d))$. Thus, $p + \dim(\mathcal{R}(Q_d)\cap\mathcal{R}(P_d)) = \operatorname{rank}Q_d \leq p$. Hence, $\operatorname{rank}Q_d = p$, and $\dim(\mathcal{R}(Q_d)\cap\mathcal{R}(P_d)) = 0$.

Next, (8) implies that

$$\mathcal{T}_d\begin{bmatrix} y_\ell\\ y_{\ell+1}\\ \vdots\\ y_{\ell+d+1}\end{bmatrix} = Q_d y_\ell + P_d\begin{bmatrix} y_{\ell+1}\\ y_{\ell+2}\\ \vdots\\ y_{\ell+d+1}\end{bmatrix} = 0.$$

Since $\dim(\mathcal{R}(Q_d)\cap\mathcal{R}(P_d)) = 0$, it follows that

$$Q_d y_\ell = P_d\begin{bmatrix} y_{\ell+1}\\ y_{\ell+2}\\ \vdots\\ y_{\ell+d+1}\end{bmatrix} = 0.$$

Since $\operatorname{rank}Q_d = p$, it follows that $y_\ell = 0$. Since $y_1 = \cdots = y_\ell = 0$, repeating the previous argument with ℓ replaced by

$\ell+1$ implies that $y_{\ell+1}=0$. By induction, it follows that, for all $k\geq\ell$, $y_k=0$. ∎

Theorem 4.6: *Let (D,N) be an LPFD of G, let $z_0\in\mathbb{C}$, and let $\overline{u}\in\mathbb{C}^m$.*

Let $y_0=\cdots=y_{\ell-1}=0$, where $\ell\triangleq\deg D$, and for all $k\geq 0$, define $u_k\triangleq z_0^k\overline{u}$ and consider

$$D(\mathbf{q})y_k=N(\mathbf{q})u_k. \tag{9}$$

Then, for all $k\geq\ell$, $y_k=0$ if and only if $N(z_0)\overline{u}=0$.

Proof: To prove sufficiency, note that Proposition 3.5 implies that, for all $k\geq 0$, $N(\mathbf{q})u_k=0$. Hence, for all $k\geq 0$, $D(\mathbf{q})y_k=0$. Since D is nonsingular, Lemma 4.5 implies that, for all $k\geq\ell$, $y_k=0$. To prove necessity, note that, for all $k\geq 0$, $N(\mathbf{q})u_k=D(\mathbf{q})y_k=0$. Therefore, Proposition 3.5 implies that $N(z_0)\overline{u}=0$. ∎

For the case of real input signals, the following result is a consequence of the sufficiency part of Theorem 4.6.

Corollary 4.7: *Let (D,N) be an LPFD of G, let $z_0\in\mathbb{C}$, and let $\overline{u}\in\mathcal{N}(N(z_0))$.*

Let $y_0=\cdots=y_{\ell-1}=0$, where $\ell\triangleq\deg D$, and for all $k\geq 0$, define $u_k\triangleq\mathrm{Re}(z_0^k\overline{u})$ and consider (9). Then, for all $k\geq\ell$, $y_k=0$.

The following result relates output zeroing in input–output models to transmission zeros of G.

Theorem 4.8: *Let (D,N) be an LPFD of G, and let z_0 be a transmission zero of G. Then $\mathcal{N}(N(z_0))\neq\{0\}$. Furthermore, let $y_0=\cdots=y_{\ell-1}=0$, where $\ell\triangleq\deg D$. Then, there exists real $u\neq 0$ such that u and $y\equiv 0$ satisfy (9).*

Proof: It follows from Proposition 3.1 that $\mathcal{N}(N(z_0))\neq\{0\}$. Let $\overline{u}$ be a nonzero vector in $\mathcal{N}(N(z_0))$ such that $\mathrm{Re}(\overline{u})\neq 0$, and define, for all $k\geq 0$, $u_k\triangleq\mathrm{Re}(z_0^k\overline{u})$. Note that $u_0=\mathrm{Re}(\overline{u})\neq 0$, and thus $u\neq 0$. Then, it follows from Corollary 4.7 that, for all $k\geq\ell$, $y_k=0$ in (9). ∎

Next, we consider the equivalence between output zeroing in input–output models and output zeroing in state-space models. In particular, Theorem 4.10 shows the equivalence between output zeroing using an MLPFD of G and output zeroing using the observable canonical form realisation of G corresponding to the MLPFD of G. The observable canonical form realisation of G obtained from an MLPFD of G is defined in Definition 4.9.

The following definition provides a MIMO extension of the observable canonical form realisation given in Polderman (1989).

Definition 4.9: Let $(D_\mathrm{M},N_\mathrm{M})$ be an MLPFD of G and write

$$D_\mathrm{M}(z)=z^\ell I+z^{\ell-1}A_1+\cdots+A_\ell, \tag{10}$$

$$N_\mathrm{M}(z)=z^\ell B_0+z^{\ell-1}B_1+\cdots+B_\ell, \tag{11}$$

where, for all $i=1,\ldots,\ell$, $A_i\in\mathbb{R}^{p\times p}$, and for all $i=0,\ldots,\ell$, $B_i\in\mathbb{R}^{p\times m}$. Let, for all $k\geq 0$,

$$\hat{x}_{k+1}=A_\mathrm{ocf}\hat{x}_k+B_\mathrm{ocf}u_k, \tag{12}$$

$$y_k=C_\mathrm{ocf}\hat{x}_k+Eu_k, \tag{13}$$

where

$$A_\mathrm{ocf}\triangleq\begin{bmatrix}0&\cdots&0&-A_\ell\\ I&\cdots&0&-A_{\ell-1}\\ \vdots&\cdots&\vdots&\vdots\\ 0&\cdots&I&-A_1\end{bmatrix}, \tag{14}$$

$$B_\mathrm{ocf}\triangleq\begin{bmatrix}B_\ell-A_\ell B_0\\ B_{\ell-1}-A_{\ell-1}B_0\\ \vdots\\ B_1-A_1B_0\end{bmatrix}, \tag{15}$$

$$C_\mathrm{ocf}\triangleq\begin{bmatrix}0&\cdots&0&I\end{bmatrix},\quad E\triangleq B_0, \tag{16}$$

$\hat{x}_k\triangleq[\,\hat{x}_{1,k}\ \cdots\ \hat{x}_{l,k}\,]^\mathrm{T}$, and for all $i=0,1,\ldots,\ell-1$,

$$\hat{x}_{\ell-i,k}\triangleq y_{k+i}+\sum_{j=1}^{i}A_jy_{k+i-j}-\sum_{j=0}^{i}B_ju_{k+i-j}. \tag{17}$$

Theorem 4.10: *Let $(D_\mathrm{M},N_\mathrm{M})$ be an MLPFD of G, let D_M and N_M be given by (10) and (11), and define $\mathcal{Z}_\mathrm{ocf}(z)\triangleq\left[\begin{smallmatrix}zI-A_\mathrm{ocf}&-B_\mathrm{ocf}\\ C_\mathrm{ocf}&E\end{smallmatrix}\right]$, where $(A_\mathrm{ocf},B_\mathrm{ocf},C_\mathrm{ocf},E)$ is the observable canonical form realisation of G corresponding to $(D_\mathrm{M},N_\mathrm{M})$ given by (14)–(16).*

Furthermore, let $\overline{u}\in\mathbb{C}^m$ and $z_0\in\mathbb{C}$.

Then, there exists $\overline{x}\in\mathbb{C}^{p\ell}$ such that $\left[\begin{smallmatrix}\overline{x}\\ \overline{u}\end{smallmatrix}\right]\in\mathcal{N}(\mathcal{Z}_\mathrm{ocf}(z_0))$ if and only if $N_\mathrm{M}(z_0)\overline{u}=0$. If these conditions hold, then

$$\overline{x}=-\begin{bmatrix}\sum_{i=0}^{\ell-1}z_0^{\ell-i-1}B_i\overline{u}\\ \sum_{i=0}^{\ell-2}z_0^{\ell-i-2}B_i\overline{u}\\ \vdots\\ B_0\overline{u}\end{bmatrix}. \tag{18}$$

Proof: To prove sufficiency, let

$$\overline{x}=-\begin{bmatrix}\sum_{i=0}^{\ell-1}z_0^{\ell-i-1}B_i\overline{u}\\ \sum_{i=0}^{\ell-2}z_0^{\ell-i-2}B_i\overline{u}\\ \vdots\\ B_0\overline{u}\end{bmatrix}.$$

Then,

$$C_{\rm ocf}\overline{x} + E\overline{u} = -B_0\overline{u} + B_0\overline{u} = 0. \tag{19}$$

Next, note that

$$\begin{aligned}
&(z_0 I - A_{\rm ocf})\overline{x} \\
&= -\begin{bmatrix} z_0 I & 0 & \cdots & 0 & A_\ell \\ -I & z_0 I & \cdots & 0 & A_{\ell-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & z_0 I + A_1 \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} \\ \sum_{i=0}^{\ell-2} z_0^{\ell-i-2} B_i \overline{u} \\ \vdots \\ B_0 \overline{u} \end{bmatrix} \\
&= \begin{bmatrix} -z_0 \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} - A_\ell B_0 \overline{u} \\ \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} - z_0 \sum_{i=0}^{\ell-2} z_0^{\ell-i-2} B_i \overline{u} - A_{\ell-1} B_0 \overline{u} \\ \vdots \\ \sum_{i=0}^{1} z_0^{1-i} B_i \overline{u} - z_0 B_0 \overline{u} - A_1 B_0 \overline{u} \end{bmatrix},
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
B_{\rm ocf}\overline{u} &= \begin{bmatrix} B_\ell - A_\ell B_0 \\ B_{\ell-1} - A_{\ell-1} B_0 \\ \vdots \\ B_1 - A_1 B_0 \end{bmatrix} \overline{u} \\
&= \begin{bmatrix} B_\ell \overline{u} - A_\ell B_0 \overline{u} \\ B_{\ell-1} \overline{u} - A_{\ell-1} B_0 \overline{u} \\ \vdots \\ B_1 \overline{u} - A_1 B_0 \overline{u} \end{bmatrix}.
\end{aligned} \tag{21}$$

Subtracting (21) from (20) yields

$$\begin{aligned}
&(z_0 I - A_{\rm ocf})\overline{x} - B_{\rm ocf}\overline{u} \\
&= \begin{bmatrix} -\sum_{i=0}^{\ell-1} z_0^{\ell-i} B_i \overline{u} - B_\ell \overline{u} \\ \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} - \sum_{i=0}^{\ell-2} z_0^{\ell-i-1} B_i \overline{u} - B_{\ell-1} \overline{u} \\ \vdots \\ \sum_{i=0}^{1} z_0^{1-i} B_i \overline{u} - z_0 B_0 \overline{u} - B_1 \overline{u} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=0}^{\ell} z_0^{\ell-i} B_i \overline{u} \\ \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} - \sum_{i=0}^{\ell-1} z_0^{\ell-i-1} B_i \overline{u} \\ \vdots \\ z_0 B_0 \overline{u} + B_1 \overline{u} - z_0 B_0 \overline{u} - B_1 \overline{u} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\ell} z_0^{\ell-i} B_i \overline{u} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned} \tag{22}$$

Since $0 = N_{\rm M}(z_0)\overline{u} = \sum_{i=0}^{\ell} z_0^{\ell-i} B_i \overline{u}$, (22) implies that

$$(z_0 I - A_{\rm ocf})\overline{x} - B_{\rm ocf}\overline{u} = 0. \tag{23}$$

It thus follows from (19) and (23) that $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}_{\rm ocf}(z_0))$.

To prove necessity, let $\overline{x} = [\,\overline{x}_1^{\rm T} \cdots \overline{x}_\ell^{\rm T}\,]^{\rm T}$, where $\overline{x}_1, \ldots, \overline{x}_\ell \in \mathbb{C}^p$. Then,

$$\begin{aligned}
0 &= (z_0 I - A_{\rm ocf})\overline{x} - B_{\rm ocf}\overline{u} \\
&= \begin{bmatrix} z_0 I & 0 & \cdots & 0 & A_\ell \\ -I & z_0 I & \cdots & 0 & A_{\ell-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & z_0 I + A_1 \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_l \end{bmatrix} \\
&\quad - \begin{bmatrix} B_\ell - A_\ell B_0 \\ B_{\ell-1} - A_{\ell-1} B_0 \\ \vdots \\ B_1 - A_1 B_0 \end{bmatrix} \overline{u}.
\end{aligned} \tag{24}$$

Note that $0 = C\overline{x} + E\overline{u} = \overline{x}_\ell + B_0\overline{u}$ implies that

$$\overline{x}_\ell = -B_0 \overline{u}. \tag{25}$$

Hence, (24) and (25) imply that

$$\begin{aligned}
\overline{x}_{\ell-1} &= (z_0 I + A_1)\overline{x}_\ell - B_1 \overline{u} + A_1 B_0 \overline{u} \\
&= -(z_0 I + A_1) B_0 \overline{u} - B_1 \overline{u} + A_1 B_0 \overline{u} \\
&= -z_0 B_0 \overline{u} - B_1 \overline{u}.
\end{aligned} \tag{26}$$

Next, (24)–(26) imply that

$$\begin{aligned}
\overline{x}_{\ell-2} &= z_0 \overline{x}_{\ell-1} + A_2 \overline{x}_\ell - B_2 \overline{u} + A_2 B_0 \overline{u} \\
&= -z_0^2 B_0 \overline{u} - z_0 B_1 \overline{u} - A_2 B_0 \overline{u} - B_2 \overline{u} + A_2 B_0 \overline{u} \\
&= -z_0^2 B_0 \overline{u} - z_0 B_1 \overline{u} - B_2 \overline{u}.
\end{aligned} \tag{27}$$

Proceeding similarly, it follows that, for all $j = 1, \ldots, \ell$, $\overline{x}_j = -\sum_{i=0}^{\ell-j} z_0^{\ell-i-j} B_i \overline{u}$, and thus (18) holds. Finally, it follows from (18) and (24) that

$$\begin{aligned}
0 &= z_0 \overline{x}_1 + A_\ell \overline{x}_\ell - B_\ell \overline{u} + A_\ell B_0 \overline{u} \\
&= -\sum_{i=0}^{\ell-1} z_0^{\ell-i} B_i \overline{u} - A_\ell B_0 \overline{u} - B_\ell \overline{u} + A_\ell B_0 \overline{u} \\
&= -\sum_{i=0}^{\ell} z_0^{\ell-i} B_i \overline{u} = -N(z_0)\overline{u}.
\end{aligned}$$

■

Note that Theorem 4.10 relates output zeroing using an MLPFD of G to output zeroing using a specific realisation of G, which is obtained from the given MLPFD and is not necessarily minimal. In order to obtain a more general result, we next consider the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realisation of G. Given an arbitrary CLPFD of a continuous-time transfer function G, Polak (1966) describes an algorithm for obtaining a minimal realisation of G. Since the algorithm in Polak (1966) is algebraic, the result holds true for discrete-time transfer functions by replacing the differentiation operator with the forward-shift operator.

For illustration, the example given in Polak (1966) is reworked in terms of $\mathbf{q}$ as Example 4.11.

Proposition 4.12 and Proposition 4.13 are consequences of the application of the algorithm in Polak (1966) to discrete-time transfer functions. Using Proposition 4.13, the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realisation of G is proved in Theorem 4.14.

Example 4.11: Let

$$G(z) = \begin{bmatrix} \dfrac{z^2-2z+3}{z^4+3z^3+7z^2+18z+6} & \dfrac{-(2z+3)(z+3)}{z^4+3z^3+7z^2+18z+6} \\ \dfrac{1}{z^2+6} & \dfrac{z+3}{z^2+6} \end{bmatrix}, \tag{28}$$

$$\hat{D}(z) = \begin{bmatrix} z^2+3z+1 & 2z+3 \\ z^3+3z^2+z & 3z^2+3z+6 \end{bmatrix},$$

$$\hat{N}(z) = \begin{bmatrix} 1 & 0 \\ z+1 & z+3 \end{bmatrix}. \tag{29}$$

Note that $G = \hat{D}^{-1}\hat{N}$, and $\deg\det\hat{D} = \operatorname{McDeg} G = 4$. Hence, Proposition 2.12 implies that $(\hat{D}, \hat{N})$ is a CLPFD of G. Let $U(z) = \left[\begin{smallmatrix} 1 & 0 \\ -z & 1 \end{smallmatrix}\right]$. Since U is unimodular, it follows from Proposition 2.8 that (D, N) is a CLPFD of G, where

$$D(z) \triangleq U(z)\hat{D}(z) = \begin{bmatrix} z^2+3z+1 & 2z+3 \\ 0 & z^2+6 \end{bmatrix},$$

$$N(z) \triangleq U(z)\hat{N}(z) = \begin{bmatrix} 1 & 0 \\ 1 & z+3 \end{bmatrix}. \tag{30}$$

In terms of the forward-shift operator, (30) has the form

$$D(\mathbf{q}) = \begin{bmatrix} \mathbf{q}^2+3\mathbf{q}+1 & 2\mathbf{q}+3 \\ 0 & \mathbf{q}^2+6 \end{bmatrix}, \quad N(\mathbf{q}) = \begin{bmatrix} 1 & 0 \\ 1 & \mathbf{q}+3 \end{bmatrix}. \tag{31}$$

Now, for all $k \ge 0$, let u_k and y_k satisfy (9), let $u_k = [\, u_{1,k} \; u_{2,k} \,]^{\mathrm{T}}$, and let $y_k = [\, y_{1,k} \; y_{2,k} \,]^{\mathrm{T}}$. Define $x_k \triangleq [\, x_{1,k} \; x_{2,k} \; x_{3,k} \; x_{4,k} \,]^{\mathrm{T}}$, where

$$x_{1,k} \triangleq y_{1,k}, \quad x_{2,k} \triangleq y_{1,k+1} + 3y_{1,k}, \tag{32}$$

$$x_{3,k} \triangleq y_{2,k}, \quad x_{4,k} \triangleq y_{2,k+1} - u_{2,k}. \tag{33}$$

Then, for all $k \ge 0$, u_k, y_k, and x_k satisfy (6) and (7) with

$$A \triangleq \begin{bmatrix} -3 & 1 & 0 & 0 \\ -1 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix},$$

$$C \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad E \triangleq 0. \tag{34}$$

It can be verified numerically that (A, B, C) is a minimal realisation of G.

The following result is a consequence of the application of the algorithm in Polak (1966) to discrete-time transfer functions.

Proposition 4.12: *Let (D, N) be a CLPFD of G and for all $k \ge 0$, let u_k and y_k satisfy (9). Then there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$, and a minimal realisation (A, B, C, E) of G such that, for all $k \ge 0$, (6) and (7) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$, where n is the McMillan degree of G.*

The following result is needed in the proof of Theorem 4.14.

Proposition 4.13: *Let (A, B, C, E) be a minimal realisation of G, let (D, N) be a CLPFD of G and, for all $k \ge 0$, let u_k and y_k satisfy (9). Then, for all $k \ge 0$, there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$ such that, for all $k \ge 0$, (6) and (7) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$, where n is the McMillan degree of G.*

Proof: Note that Proposition 4.12 implies that there exist $\overline{L}_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $\overline{L}_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$, and a minimal realisation $(\overline{A}, \overline{B}, \overline{C}, \overline{E})$ of G such that, for all $k \ge 0$, (6) and (7) hold with A, B, C, E, and x_k replaced with $\overline{A}, \overline{B}, \overline{C}, \overline{E}$, and $\overline{x}_k$, respectively, and $\overline{x}_k \triangleq \overline{L}_u(\mathbf{q})u_k + \overline{L}_y(\mathbf{q})y_k$. Next, Proposition 16.9.8 in Bernstein (2018, p. 1272) implies that there exists a unique nonsingular $S \in \mathbb{R}^{n\times n}$ such that $A = S\overline{A}S^{-1}$, $B = S\overline{B}$, and $C = \overline{C}S^{-1}$. Define $L_u \triangleq S\overline{L}_u$ and $L_y \triangleq S\overline{L}_y$. Hence, for all $k \ge 0$, (6) and (7) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$. ■

The following result establishes the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realisation of G. Note that, unlike Theorem 4.10, the LPFD in the following is coprime but not necessarily monic.

Theorem 4.14: *Let (D, N) be a CLPFD of G, let (A, B, C, E) be an nth-order minimal realisation of G, $z_0 \in \mathbb{C}$, and let $\overline{u} \in \mathbb{C}^m$.*

Then, there exists $\overline{x} \in \mathbb{C}^n$ such that $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$ if and only if $N(z_0)\overline{u} = 0$.

Proof: Suppose that, for all $k \ge 0$, u_k and y_k satisfy (9). Then Proposition 4.13 implies that, for all $k \ge 0$, there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$ and $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$ such that, for all $k \ge 0$, (6) and (7) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$. For all $k \ge 0$, define $u_k \triangleq z_0^k\overline{u}$. To prove necessity, define $x_0 \triangleq \overline{x}$. Then, statement (i) in Proposition 4.2 implies that, for all $k \ge 0$, $y_k = 0$.

Hence, it follows from Theorem 4.6 that $N(z_0)\overline{u} = 0$. To prove sufficiency, define $\overline{x} \triangleq L_u(z_0)\overline{u}$, write $L_y(\mathbf{q}) = q^r P_r + \cdots + qP_1 + P_0$, define $\ell \triangleq \deg D$, and suppose that $y_0 = y_1 = \cdots = y_c = 0$, where $c \triangleq \max\{r, \ell - 1\}$. Note that $x_k = L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k = L_u(\mathbf{q})z_0^k\overline{u} + L_y(\mathbf{q})y_k = L_u(z_0)z_0^k\overline{u} + L_y(\mathbf{q})y_k$. Hence $x_0 = L_u(z_0)\overline{u} + P_r y_r + \cdots + P_1 y_1 + P_0 y_0 = L_u(z_0)\overline{u} = \overline{x}$. Next, it follows from Theorem 4.6 that, for all $k \geq 0, y_k = 0$. Hence statement (ii) in Proposition 4.2 implies that $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$. ■

In the case where z_0 is a transmission zero of G, Theorem 4.4 implies that there exists nonzero $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$, where $\overline{x} \neq 0$ and $\overline{u} \neq 0$, and thus Theorem 4.14 implies that $N(z_0)\overline{u} = 0$. Therefore, there exist $\overline{x} \neq 0$ and $\overline{u} \neq 0$ such that $\left[\begin{smallmatrix} \overline{x} \\ \overline{u} \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}(z_0))$ and $N(z_0)\overline{u} = 0$.

The following example illustrates the equivalence between output zeroing in input–output models and output zeroing in state-space models due to transmission zeros.

Example 4.15: Consider the discrete-time transfer function

$$G(z) = \begin{bmatrix} \dfrac{z-3}{z+2} & 0 \\ \dfrac{1}{z+1} & \dfrac{z}{z+1} \\ 1 & \dfrac{9}{z} \end{bmatrix} \quad (35)$$

and consider the minimal realisation of G given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad (36)$$

$$C = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (37)$$

Since $\operatorname{rank} \mathcal{Z}(3) = 4 < 5 = \operatorname{rank} \mathcal{Z}$, it follows that $z = 3$ is a transmission zero of G. For all $k \geq 0$, let u_k, y_k, and x_k satisfy (6) and (7). Then, it follows from Proposition 4.2 that, if

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(3)), \quad (38)$$

and for all $k \geq 1$, $u_k = 3^k u_0$, then $y \equiv 0$. For example, noting

$$\begin{bmatrix} 3/5 \\ 1 \\ -1/3 \\ 3 \\ -1 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(3)), \quad (39)$$

it follows that $y \equiv 0$ with $x_0 = \left[\begin{smallmatrix} 3/5 \\ 1 \\ -1/3 \end{smallmatrix}\right]$ and, for all $k \geq 0$, $u_k = 3^k \left[\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right]$.

Next, taking the Z-transform of (6) and (7) yields

$$\hat{y}(z) = G(z)\hat{u}(z) + zC(zI - A)^{-1}x_0, \quad (40)$$

where $\hat{u}$ and $\hat{y}$ denote the Z-transforms of u and y, respectively. Note that (40) includes separate terms for the free response and the forced response of (6) and (7).

An alternative time-domain representation of (6) and (7) can be obtained by replacing z by the forward-shift operator $\mathbf{q}$.

To do this, we first factor $G(z) = D(z)^{-1}N(z)$, where

$$D(z) = zI_3 + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (41)$$

$$N(z) = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & 9 \end{bmatrix}. \quad (42)$$

Note that (D, N) is an MLPFD of G. Then, for all $k \geq 0$, (6) and (7) have the equivalent time-domain input–output model (9), where

$$D(\mathbf{q}) = \begin{bmatrix} \mathbf{q}+2 & 0 & 0 \\ 0 & \mathbf{q}+1 & 0 \\ 0 & 0 & \mathbf{q} \end{bmatrix}, \quad N(\mathbf{q}) = \begin{bmatrix} \mathbf{q}-3 & 0 \\ 1 & \mathbf{q} \\ \mathbf{q} & 9 \end{bmatrix}. \quad (43)$$

Note that the free response $zC(zI - A)^{-1}x_0$ in (40) has no counterpart in (9). In fact, the response of (9) includes both the free response and the forced response (Aljanaideh & Bernstein, 2018). Now, in (9), letting $y_0 = 0$ and, for all $k \geq 0$, $u_k = 3^k \left[\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right]$ yields $y \equiv 0$. Furthermore, note that $u_0 \in \mathcal{N}(N(3))$.

Hence, the input that produces identically zero output is obtained from the Rosenbrock matrix $\mathcal{Z}$ for a state-space model as well as the numerator polynomial matrix N for an input–output model.

To further illustrate the connection between output zeroing in input–output models and output zeroing in state-space models, we consider the observable canonical form realisation $(A_{\rm ocf}, B_{\rm ocf}, C_{\rm ocf}, E)$ of G corresponding to (D, N), where

$$A_{\rm ocf} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{\rm ocf} = \begin{bmatrix} -5 & 0 \\ 1 & -1 \\ 0 & -9 \end{bmatrix}, \quad C_{\rm ocf} = I_3. \quad (44)$$

By using (17), the state of the observable canonical form realisation (12) and (13) can be constructed in terms of input and output data. For this example, the initial condition obtained from (17) is given by

$$\hat{x}_0 = y_0 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_0. \quad (45)$$

Now, setting $y_0 = 0$ and $u_0 = \left[\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right]$ in (45) yields $\hat{x}_0 = \left[\begin{smallmatrix} -3 \\ 1 \\ -3 \end{smallmatrix}\right]$, which is the initial condition that, along with the output-zeroing input, produces the identically zero output. It can be verified numerically that $\left[\begin{smallmatrix} \hat{x}_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{N}(\mathcal{Z}_{\rm ocf}(3))$, where $\mathcal{Z}_{\rm ocf}(z) \triangleq \left[\begin{smallmatrix} zI - A_{\rm ocf} & -B_{\rm ocf} \\ C_{\rm ocf} & E \end{smallmatrix}\right]$. Hence, the vector consisting of this initial condition and the initial input lies in the null space of the Rosenbrock system matrix.

5. Conclusions

This paper explored the properties of zero dynamics within the context of input–output models. In particular, the transmission zeros were characterised using left polynomial fraction description input–output models, the zero dynamics of these models were defined, and the solutions of the zero dynamics of these models that correspond to the transmission zeros of the system were described. Furthermore, the equivalence between output zeroing in input–output models and output zeroing in state-space models was discussed. Finally, an example is given to illustrate the results on output zeroing.

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