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Decomposition of the Retrospective Performance Variable in Adaptive Input Estimation

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Abstract— **Retrospective cost input estimation (RCIE) is an adaptive input estimation technique that is based on the minimization of a retrospective performance variable using recursive least squares. In this paper, in order to obtain a better understanding of the underlying mechanism and the performance of RCIE, a decomposition of the retrospective performance variable into the sum of a performance term and a model-matching term is presented. Since this decomposition involves time-varying input-output models, the construction of LTV state space realizations from LTV input-output models as well as the construction of LTV input-output models from LTV state space models are presented. Analysis of the decomposition shows how RCIE avoids convergence to an estimator that is destabilizing or has poor performance. A numerical example is used to illustrate the derived results and observations.**

I. INTRODUCTION

In classical state estimation, the main objective is to estimate the unmeasured states, taking advantage of knowledge of the system dynamics as well as the process- and sensor-noise statistics. It is often the case in practice, however, that, in addition to stochastic inputs, the system has a deterministic input. If this signal is known, then, in the spirit of the separation principle, it can be replicated in the estimator. If, however, this signal is unknown, then it is of interest to obtain state estimates that are unbiased. Approaches to this problem include unbiased Kalman filters, unknown input observers, and sliding-mode observers [1]–[8].

An alternative approach, known as *input estimation* or *input reconstruction*, is to estimate the unknown input and replicate the estimated input in the state estimator. Input estimation has not been as widely studied as state estimation, but interest in this problem has grown steadily over the last few decades [9]–[24]. In addition to providing more accurate state estimates than other methods, input estimation yields an estimate of the unknown input that is useful for applications such as sensor/actuator health assessment, analysis of exogenous disturbances, and tracking and navigation [25]–[27].

The present paper focuses on retrospective cost input estimation (RCIE) developed in [23]. RCIE is an adaptive input estimation technique based on a retrospective performance variable, which depends on a target model that is based on the closed loop system dynamics. By employing retrospective cost optimization based on recursive least squares (RLS) to update the coefficients of the input estimator, RCIE replicates the estimated input in the Kalman filter and adapts the estimator coefficients by using the innovations as the error metric. As a special case, this technique was applied to single- and double-integrator dynamics to estimate the velocity and acceleration of a maneuvering vehicle.

The goal of the present paper is to investigate the underlying mechanism and performance of RCIE. In this direction, this paper provides a detailed analysis of the decomposition of the retrospective performance variable, which provides insight into the achievable performance of RCIE. In particular, the retrospective performance variable is decomposed into the sum of a performance term and a model-matching term. The performance term consists of a closed-loop time-domain transfer function, whereas the model-matching term involves a closed-loop time-domain transfer function and the target model, both driven by the virtual external input perturbation. This work is motivated by the decomposition of the retrospective performance variable given in [28] within the context of retrospective cost adaptive control. However, unlike [28], the system dynamics and target model in the present paper are linear time-varying (LTV), and hence the approach given in [28] is not applicable here.

The main contribution of the present paper is thus the development of an alternative approach to the decomposition of the retrospective performance variable that is applicable to LTV models. This approach depends on the construction of LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models. The existing results on LTV input-output models in [29]–[34] are presented in terms of abstract input-output maps and infinite power series, and thus are not directly implementable. Consequently, the present paper gives simple and easily implementable algebraic results on LTV input-output dynamics needed to derive the decomposition of the retrospective performance variable in RCIE.

The numerical example given in the present paper shows that RCIE drives the retrospective performance approximately to zero, where the performance and model-matching terms, which are sign indefinite, are not necessarily small but have approximately equal magnitudes and opposite signs. Furthermore, it is shown that, as the estimator converges, the virtual external input perturbation converges to zero, and thus the model-matching and performance terms converge to zero. The retrospective performance variable decomposition thus shows how RCIE avoids convergence to an estimator that is destabilizing or has poor performance.

II. RETROSPECTIVE COST INPUT ESTIMATION

Consider the LTV discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + G_k d_k + D_{1,k} w_k, \qquad (1)$$

$$y_k = C_k x_k + D_{2,k} v_k, \qquad (2)$$

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where $x_k \in \mathbb{R}^{l_x}$ is the unknown state, $u_k \in \mathbb{R}^{l_u}$ is the known input, $d_k \in \mathbb{R}^{l_d}$ is the unknown input, $w_k \in \mathbb{R}^{l_w}$ is the unknown standard Gaussian white process noise, $y_k \in \mathbb{R}^{l_y}$ is the measured output, and $v_k \in \mathbb{R}^{l_v}$ is the unknown standard Gaussian white measurement noise. The matrices A_k, B_k, G_k, $D_{1,k}$, C_k, and $D_{2,k}$ are assumed to be known. Define $V_{1,k} \triangleq D_{1,k}D_{1,k}^{\mathrm{T}}$ and $V_{2,k} \triangleq D_{2,k}D_{2,k}^{\mathrm{T}}$. The goal is to estimate d_k and x_k.

A. Input Estimation

Consider the Kalman filter forecast step

$$x_{\mathrm{fc},k+1} = A_k x_{\mathrm{da},k} + B_k u_k + G_k \hat{d}_k, \tag{3}$$

$$y_{\mathrm{fc},k} = C_k x_{\mathrm{fc},k}, \quad z_k = y_{\mathrm{fc},k} - y_k, \tag{4}$$

where $\hat{d}_k \in \mathbb{R}^{l_d}$ is the input estimate, $x_{\mathrm{da,k}} \in \mathbb{R}^{l_x}$ is the data-assimilation state, $x_{\mathrm{fc},k} \in \mathbb{R}^{l_x}$ is the forecast state, $z_k \in \mathbb{R}^{l_y}$ is the innovations, and $x_{\mathrm{fc},0} = 0$. The input estimate $\hat{d}_k$ is obtained as the output of the *input-estimation subsystem* of order n_c given by

$$\hat{d}_k = \sum_{i=1}^{n_\mathrm{c}} P_{i,k}\hat{d}_{k-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k} z_{k-i}, \tag{5}$$

where $P_{i,k} \in \mathbb{R}^{l_d \times l_d}$ and $Q_{i,k} \in \mathbb{R}^{l_d \times l_y}$. RCIE minimizes z_{k+1} by updating $P_{i,k}$ and $Q_{i,k}$. The subsystem (5) can be reformulated as

$$\hat{d}_k = \Phi_k \theta_k, \tag{6}$$

where the regressor matrix Φ_k is defined by

$$\Phi_k \triangleq \begin{bmatrix} \hat{d}_{k-1} & \cdots & \hat{d}_{k-n_\mathrm{c}} & z_k & \cdots & z_{k-n_\mathrm{c}} \end{bmatrix} \otimes I_{l_d} \in \mathbb{R}^{l_d \times l_\theta}, \tag{7}$$

the coefficient vector θ_k is defined by

$$\theta_k \triangleq \mathrm{vec} \begin{bmatrix} P_{1,k} & \cdots & P_{n_\mathrm{c},k} & Q_{0,k} & \cdots & Q_{n_\mathrm{c},k} \end{bmatrix} \in \mathbb{R}^{l_\theta}, \tag{8}$$

and $l_\theta \triangleq l_d^2 n_\mathrm{c} + l_d l_y(n_\mathrm{c}+1)$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator. In terms of the forward shift operator $\mathbf{q}$, (5) can be written as

$$\hat{d}_k = G_{\hat{d}z,k}(\mathbf{q}) z_k, \tag{9}$$

where

$$G_{\hat{d}z,k}(\mathbf{q}) \triangleq D_{\hat{d}z,k}^{-1}(\mathbf{q}) N_{\hat{d}z,k}(\mathbf{q}), \tag{10}$$

$$D_{\hat{d}z,k}(\mathbf{q}) \triangleq I_{l_d}\mathbf{q}^{n_\mathrm{c}} - P_{1,k}\mathbf{q}^{n_\mathrm{c}-1} - \cdots - P_{n_\mathrm{c},k}, \tag{11}$$

$$N_{\hat{d}z,k}(\mathbf{q}) \triangleq Q_{0,k}\mathbf{q}^{n_\mathrm{c}} + Q_{1,k}\mathbf{q}^{n_\mathrm{c}-1} + \cdots + Q_{n_\mathrm{c},k}. \tag{12}$$

Next, define the filtered signals

$$\Phi_{\mathrm{f},k} \triangleq G_{\mathrm{f},k}(\mathbf{q})\Phi_k, \quad \hat{d}_{\mathrm{f},k} \triangleq G_{\mathrm{f},k}(\mathbf{q})\hat{d}_k. \tag{13}$$

Note that $G_{\mathrm{f},k}$ is a filter of window length $n_\mathrm{f} \geq 1$. Further details of the filter $G_{\mathrm{f},k}$ are given in the subsection II-C. Define the *retrospective performance variable*

$$z_{\mathrm{rc},k}(\hat{\theta}) \triangleq z_k - \left(\hat{d}_{\mathrm{f},k} - \Phi_{\mathrm{f},k}\hat{\theta}\right), \tag{14}$$

where the coefficient vector $\hat{\theta} \in \mathbb{R}^{l_\theta}$ denotes a variable for optimization. The retrospective performance variable $z_{\mathrm{rc},k}(\hat{\theta})$ is used to determine the updated coefficient vector θ_{k+1} by minimizing a function of $z_{\mathrm{rc},k}(\hat{\theta})$. The optimized value of $z_{\mathrm{rc},k}$ is thus given by

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_k - \left(\hat{d}_{\mathrm{f},k} - \Phi_{\mathrm{f},k}\theta_{k+1}\right). \tag{15}$$

Next, define the retrospective cost function

$$J_k(\hat{\theta}) \triangleq \sum_{i=0}^{k} \lambda^{k-i}\left(z_{\mathrm{rc},i}^{\mathrm{T}}(\hat{\theta}) R_z z_{\mathrm{rc},i}(\hat{\theta}) + \hat{\theta}^{\mathrm{T}}\Phi_i^{\mathrm{T}} R_d \Phi_i \hat{\theta}\right) + \lambda^{k+1}(\hat{\theta}-\theta_0)^{\mathrm{T}} R_\theta (\hat{\theta}-\theta_0), \tag{16}$$

where $R_z \in \mathbb{R}^{l_y \times l_y}$ and $R_\theta \in \mathbb{R}^{l_\theta \times l_\theta}$ are positive definite, $R_d \in \mathbb{R}^{l_d \times l_d}$ is positive semi-definite, and $\lambda \in (0,1]$ is the forgetting factor. Define $P_0 \triangleq R_\theta^{-1}$. Then, for all $k \geq 0$, the cumulative cost function $J_k(\hat{\theta})$ has the unique global minimizer θ_{k+1} given by the RLS update

$$P_{k+1} = \frac{1}{\lambda}(P_k - P_k\widetilde{\Phi}_k^{\mathrm{T}}\Gamma_k\widetilde{\Phi}_k P_k), \tag{17}$$

$$\theta_{k+1} = \theta_k - P_k\widetilde{\Phi}_k^{\mathrm{T}}\Gamma_k(\widetilde{z}_k + \widetilde{\Phi}_k\theta_k), \tag{18}$$

$$\Gamma_k \triangleq (\lambda\widetilde{R}^{-1} + \widetilde{\Phi}_k P_k \widetilde{\Phi}_k^{\mathrm{T}})^{-1}, \quad \widetilde{\Phi}_k \triangleq \begin{bmatrix}\Phi_{\mathrm{f},k} \\ \Phi_k\end{bmatrix}, \tag{19}$$

$$\widetilde{R} \triangleq \begin{bmatrix} R_z & 0 \\ 0 & R_d \end{bmatrix}, \quad \widetilde{z}_k \triangleq \begin{bmatrix} z_k - \hat{d}_{\mathrm{f},k} \\ 0_{l_d \times 1}\end{bmatrix}. \tag{20}$$

Using the updated coefficient vector given by (18), the estimated input at step $k+1$ is given by replacing k by $k+1$ in (6). We choose $\theta_0 = 0$, and thus $\hat{d}_0 = 0$.

B. State Estimation

In order to estimate the state x_k, $x_{\mathrm{fc},k}$ given by (3) is used to obtain the estimate $x_{\mathrm{da},k}$ of x_k given by the Kalman filter data-assimilation step

$$x_{\mathrm{da},k} = x_{\mathrm{fc},k} + K_{\mathrm{da},k} z_k, \tag{21}$$

where the state estimator gain $K_{\mathrm{da},k} \in \mathbb{R}^{l_x \times l_y}$ is given by

$$K_{\mathrm{da},k} = -P_{\mathrm{f},k} C_k^{\mathrm{T}}(C_k P_{\mathrm{f},k} C_k^{\mathrm{T}} + V_{2,k})^{-1}, \tag{22}$$

the data-assimilation error covariance $P_{\mathrm{da},k} \in \mathbb{R}^{l_x \times l_x}$ and the forecast error covariance $P_{\mathrm{f},k} \in \mathbb{R}^{l_x \times l_x}$ are given by

$$P_{\mathrm{da},k} = (I + K_{\mathrm{da},k}C_k)P_{\mathrm{f},k}, \tag{23}$$

$$P_{\mathrm{f},k+1} = A_k P_{\mathrm{da},k} A_k^{\mathrm{T}} + V_{1,k} + \widetilde{V}_k, \tag{24}$$

where $\widetilde{V}_k \triangleq G_k \text{ var } (d_k - \hat{d}_k)G_k^{\mathrm{T}} + A_k \text{ cov } (x_k - x_{\mathrm{da},k}, d_k - \hat{d}_k)G_k^{\mathrm{T}} + G_k \text{ cov } (x_k - x_{\mathrm{da},k}, d_k - \hat{d}_k)A_k^{\mathrm{T}}$, and $P_{\mathrm{f},0} = 0$.

C. The Filter G_f

We choose $G_{\mathrm{f},k}(\mathbf{q})$ to be the FIR filter

$$G_{\mathrm{f},k}(\mathbf{q}) = \sum_{i=1}^{n_\mathrm{f}} H_{i,k}\frac{1}{\mathbf{q}^i}, \tag{25}$$

where, for all $k \geq 0$,

$$H_{i,k} \triangleq \begin{cases} C_k G_{k-1}, & k \geq i = 1, \\ C_k\left(\prod_{j=1}^{i-1}\overline{A}_{k-j}\right)G_{k-i}, & k \geq i \geq 2, \\ 0, & i > k, \end{cases} \tag{26}$$

$$\overline{A}_k \triangleq A_k(I + K_{\mathrm{da},k}C_k). \tag{27}$$

This particular choice for the filter was given in [23] and is observed to be effective in the successful implementation of the RCIE algorithm.

III. RELATION BETWEEN LTV INPUT-OUTPUT MODELS AND LTV STATE SPACE MODELS

This section gives the construction of LTV state space realizations for LTV input-output models and the construction of LTV input-output models for LTV state space models. The proofs of these results are omitted due to space limitations.

Definition 3.1: Consider the LTV state space model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{28}$$
$$y_k = C_k x_k + E_k u_k, \tag{29}$$

where, $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the input, and $y_k \in \mathbb{R}^p$ is the output. Define the *observability matrix*

$$\mathcal{O}_k \triangleq \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \\ C_{k+n-1}A_{k+n-2}\dots A_{k+1}A_k \end{bmatrix}. \tag{30}$$

Assume that, for all $k \geq 0$, $\operatorname{rank} \mathcal{O}_k = n$. Then, (28) and (29) is *completely observable*.

Definition 3.2: Let $G_k(\mathbf{q})$ be the time-domain transfer function of an LTV system at step k. Assume that, for all $k \geq 0$, $G_k(\mathbf{q}) = D_k^{-1}(\mathbf{q}) N_k(\mathbf{q})$, where

$$D_k(\mathbf{q}) \triangleq \mathbf{q}^n + D_{1,k}\mathbf{q}^{n-1} + \cdots + D_{n,k}, \tag{31}$$
$$N_k(\mathbf{q}) \triangleq N_{0,k}\mathbf{q}^n + N_{1,k}\mathbf{q}^{n-1} + \cdots + N_{n,k}, \tag{32}$$

and, for all $k \geq 0$, $D_{1,k}, \dots, D_{n,k} \in \mathbb{R}^{p\times p}$ and $N_{0,k}, \dots, N_{n,k} \in \mathbb{R}^{p\times m}$. Then, for all $k \geq 0$,

$$\begin{aligned} y_{k+n} + D_{1,k}y_{k+n-1} + \cdots + D_{n,k}y_k \\ = N_{0,k}u_{k+n} + \cdots + N_{n,k}u_k \end{aligned} \tag{33}$$

is an *input-output model* of $G_k(\mathbf{q})$.

Proposition 3.3: Assume that (28) and (29) is completely observable. Then, for all $k \geq 0$, an input-output model corresponding to (28) and (29) is given by (33) where,

$$N_{i,k} = \begin{cases} \overline{H}_{0,k+n}, & i = 0, \\ \overline{H}_{i,k+n-i} + \sum_{j=0}^{i-1} D_{i-j,k}\overline{H}_{j,k+n-i}, & 1 \leq i \leq n, \end{cases} \tag{34}$$

$$\begin{bmatrix} D_{n,k} & \cdots & D_{1,k} \end{bmatrix} = -C_{k+n}\Psi_{n,0,k}\mathcal{O}_k^{\mathrm{L}}, \tag{35}$$

$$\overline{H}_{i,k} \triangleq \begin{cases} E_k, & i = 0, \\ C_{k+i}\Psi_{i,1,k}B_k, & i \geq 1, \end{cases} \tag{36}$$

$$\Psi_{i,j,k} \triangleq \begin{cases} A_{k+i-1}A_{k+i-2}\dots A_{k+j}, & i > j, \\ I, & i = j, \\ 0, & i < j, \end{cases} \tag{37}$$

and $\mathcal{O}_k^{\mathrm{L}}$ is a left inverse of $\mathcal{O}_k$ defined in (30).

Proposition 3.4: A state space model corresponding to the input-output model in (33) is given by (28) and (29), where, for all $k \geq 0$,

$$A_k = \begin{bmatrix} 0 & \cdots & 0 & -D_{n,k} \\ I & \cdots & 0 & -D_{n-1,k-1} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & -D_{1,k-n+1} \end{bmatrix}, \tag{38}$$

$$B_k = \begin{bmatrix} N_{n,k} - D_{n,k}N_{0,k-n} \\ N_{n-1,k-1} - D_{n-1,k-1}N_{0,k-n} \\ \vdots \\ N_{1,k-n+1} - D_{1,k-n+1}N_{0,k-n} \end{bmatrix}, \tag{39}$$

$$C_k = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}, \quad E_k = N_{0,k-n}, \tag{40}$$

and, for all $k < 0$, $D_{1,k} = \cdots = D_{n,k} = N_{0,k} = \cdots = N_{n,k} = 0$. Furthermore, $x_k = \begin{bmatrix} x_{1,k}^{\mathrm{T}} & x_{2,k}^{\mathrm{T}} & \cdots & x_{n,k}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where, for all $i = 1, \dots, n$,

$$\begin{aligned} x_{i,k} = y_{k+n-i} + \sum_{j=1}^{n-i} D_{j,k-i}y_{k+n-i-j} \\ - \sum_{j=0}^{n-i} N_{j,k-i}u_{k+n-i-j}. \end{aligned} \tag{41}$$

IV. DECOMPOSITION OF THE RETROSPECTIVE PERFORMANCE VARIABLE

This section shows that the retrospective performance variable can be decomposed into the sum of a performance term and a model-matching term.

Definition 4.1: Let $D_{1,k}, \dots, D_{n,k} \in \mathbb{R}^{p\times p}$, let $N_{0,k}, \dots, N_{n,k} \in \mathbb{R}^{p\times m}$, let $y_{k-n}, \dots, y_{-1} \in \mathbb{R}^p$ be initial output data, let $(\theta_k)_{k=-n}^{\infty} \in \mathbb{R}^r$, and, for all $k \geq -n$, let $u_k : \mathbb{R}^r \to \mathbb{R}^m$. Then, the *FIA sequence* $(y_k(\theta_k))_{k=0}^{\infty}$ is given by the *fixed-input-argument (FIA) filter*

$$\begin{aligned} y_k(\theta_k) + D_{1,k}y_{k-1}(\theta_{k-1}) + \cdots + D_{n,k}y_{k-n}(\theta_{k-n}) \\ = N_{0,k}u_k(\theta_k) + \cdots + N_{n,k}u_{k-n}(\theta_k), \end{aligned} \tag{42}$$

where, for all $k \in [-n, -1], y_k(\theta_k) \triangleq y_k$.

Note that, in (42), at each step k, the arguments of $u_{k-n}, \dots, u_k$ are fixed at the current value θ_k. In contrast, the left hand side defines the current output $y_k(\theta_k)$ which depends on the past output values $y_{k-n}(\theta_{k-n}), \dots, y_{k-1}(\theta_{k-1})$. For convenience, (42) is written as either $D_k(\mathbf{q})y_k(\theta_k) = N_k(\mathbf{q})u_k(\theta_{\overline{k}})$ or $y_k(\theta_k) = G_k(\mathbf{q})u_k(\theta_{\overline{k}})$, where $G_k(\mathbf{q}) \triangleq D_k^{-1}(\mathbf{q})N_k(\mathbf{q})$.

Define the *virtual external input perturbation*

$$\widetilde{d}_k(\theta_{k+1}) \triangleq \hat{d}_{k+n_c} - \Phi_{k+n_c}\theta_{k+1}. \tag{43}$$

Let $\widetilde{d}_{\mathrm{f},k}(\theta_{k+1})$ be given by the FIA filter

$$\widetilde{d}_{\mathrm{f},k}(\theta_{k+1}) = \overline{G}_{\mathrm{f},k}(\mathbf{q})\widetilde{d}_k(\theta_{\overline{k+1}}), \tag{44}$$

where $\overline{G}_{\mathrm{f},k}(\mathbf{q}) \triangleq \mathbf{q}^{-n_c}G_{\mathrm{f},k}(\mathbf{q})$. Using (44), (15) can be written as

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_k - \widetilde{d}_{\mathrm{f},k}(\theta_{k+1}). \tag{45}$$

The following matrices are used in Theorem 4.1.

$$\hat{A}_k \triangleq \begin{bmatrix} 0 & \cdots & 0 & P_{n_\mathrm{c},k+1} \\ I & \cdots & 0 & P_{n_\mathrm{c}-1,k} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & P_{1,k-n_\mathrm{c}+2} \end{bmatrix}, \hat{G} \triangleq \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{l_d n_\mathrm{c} \times l_d}, \tag{46}$$

$$\hat{B}_k \triangleq \begin{bmatrix} Q_{n_\mathrm{c},k+1} + P_{n_\mathrm{c},k+1} Q_{0,k-n_\mathrm{c}+1} \\ Q_{n_\mathrm{c}-1,k} + P_{n_\mathrm{c}-1,k} Q_{0,k-n_\mathrm{c}+1} \\ \vdots \\ Q_{1,k-n_\mathrm{c}+2} + P_{1,k-n_\mathrm{c}+2} Q_{0,k-n_\mathrm{c}+1} \end{bmatrix}, \tag{47}$$

$$\hat{C} \triangleq \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix} \in \mathbb{R}^{l_d \times l_d n_\mathrm{c}}, \hat{D}_k \triangleq Q_{0,k-n_\mathrm{c}+1}, \tag{48}$$

$$B_{\mathrm{a},k} \triangleq \begin{bmatrix} G_k & D_{1,k} & -\overline{B}_k D_{2,k} \end{bmatrix}, \overline{B}_k \triangleq -A_k K_{\mathrm{da},k}, \tag{49}$$

$$D_{\mathrm{a},k} \triangleq \begin{bmatrix} 0_{l_y \times l_d} & 0_{l_y \times l_w} & -D_{2,k} \end{bmatrix}, \tag{50}$$

$$\widetilde{A}_k \triangleq \begin{bmatrix} \hat{A}_k & -\hat{B}_k C_k \\ -G_k \hat{C} & \overline{A}_k + G_k \hat{D}_k C_k \end{bmatrix}, \widetilde{C}_k \triangleq \begin{bmatrix} 0 & -C_k \end{bmatrix}, \tag{51}$$

$$\widetilde{B}_k \triangleq \begin{bmatrix} \hat{G} & \hat{B}_k D_{\mathrm{a},k} \\ 0 & B_{\mathrm{a},k} - G_k \hat{D}_k D_{\mathrm{a},k} \end{bmatrix}, \widetilde{D}_k \triangleq \begin{bmatrix} 0 & D_{\mathrm{a},k} \end{bmatrix}. \tag{52}$$

The following result presents the *retrospective performance variable decomposition*, which shows that the retrospective performance variable is a combination of the closed-loop performance and the extent to which the updated closed-loop transfer function from $\widetilde{d}_k(\theta_{k+1})$ to z_k matches the filter $\overline{G}_{\mathrm{f},k}(\mathbf{q})$. Henceforth, $\overline{G}_{\mathrm{f},k}(\mathbf{q})$ is called the *target model* since it serves as the target for the closed-loop transfer function from $\widetilde{d}_k(\theta_{k+1})$ to z_k.

Theorem 4.1: For all $k \geq 0$,

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_{\mathrm{pp},k}(\theta_{k+1}) + z_{\mathrm{mm},k}(\theta_{k+1}), \tag{53}$$

where the performance term $z_{\mathrm{pp},k}(\theta_{k+1})$ and the model-matching term $z_{\mathrm{mm},k}(\theta_{k+1})$ are defined as

$$z_{\mathrm{pp},k}(\theta_{k+1}) \triangleq G_{z\overline{u},k}(\mathbf{q}) \overline{u}_k, \tag{54}$$

$$z_{\mathrm{mm},k}(\theta_{k+1}) \triangleq G_{z\widetilde{d},k}(\mathbf{q}) \widetilde{d}_k(\theta_{k+1}) - \overline{G}_{\mathrm{f},k}(\mathbf{q}) \widetilde{d}_k(\theta_{\overline{k+1}}), \tag{55}$$

and $\overline{u}_k \triangleq \begin{bmatrix} d_k^\mathrm{T} & w_k^\mathrm{T} & v_k^\mathrm{T} \end{bmatrix}^\mathrm{T}$. The time-domain transfer functions $G_{z\overline{u},k}$ and $G_{z\widetilde{d},k}$ are given by

$$\begin{bmatrix} G_{z\widetilde{d},k} & G_{z\overline{u},k} \end{bmatrix} \triangleq G_{z\widetilde{u},k}, \tag{56}$$

where $G_{z\widetilde{u},k}$ is the time-domain transfer function of the system represented by the state space model

$$\widetilde{x}_{k+1} = \widetilde{A}_k \widetilde{x}_k + \widetilde{B}_k \widetilde{u}_k, \tag{57}$$

$$z_k = \widetilde{C}_k \widetilde{x}_k + \widetilde{D}_k \widetilde{u}_k, \tag{58}$$

$\widetilde{u}_k \triangleq \begin{bmatrix} \widetilde{d}_k^\mathrm{T}(\theta_{k+1}) & \overline{u}_k^\mathrm{T} \end{bmatrix}^\mathrm{T}$, $\widetilde{x}_0 \triangleq \begin{bmatrix} \hat{d}_{n_\mathrm{c}-1}^\mathrm{T} & \cdots & \hat{d}_0^\mathrm{T} & x_0^\mathrm{T} \end{bmatrix}^\mathrm{T}$, and $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{D}_k$ are defined in (51) and (52).

Proof: Note that (2) and (4) imply that

$$z_k = -C_k e_{\mathrm{fc},k} - D_{2,k} v_k, \tag{59}$$

where $e_{\mathrm{fc},k} \triangleq x_k - x_{\mathrm{fc},k}$. Note that it follows from (1)–(4) and (21) that

$$e_{\mathrm{fc},k+1} = \overline{A}_k e_{\mathrm{fc},k} + G_k(d_k - \hat{d}_k) + D_{1,k} w_k - \overline{B}_k D_{2,k} v_k, \tag{60}$$

where $\overline{A}_k$ and $\overline{B}_k$ are defined in (27) and (49) respectively. Then, (60) and (59) can be written as

$$e_{\mathrm{fc},k+1} = \overline{A}_k e_{\mathrm{fc},k} + B_{\mathrm{a},k} \overline{u}_k - G_k \hat{d}_k, \tag{61}$$

$$z_k = -C_k e_{\mathrm{fc},k} + D_{\mathrm{a},k} \overline{u}_k, \tag{62}$$

where $B_{\mathrm{a},k}$ and $D_{\mathrm{a},k}$ are defined in (49) and (50). Next, it follows from (5) that

$$\Phi_{k+n_\mathrm{c}} \theta_{k+1} = \sum_{i=1}^{n_\mathrm{c}} P_{i,k+1} \hat{d}_{k+n_\mathrm{c}-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k+1} z_{k+n_\mathrm{c}-i},$$

which when substituted in (43) yields

$$\hat{d}_{k+n_\mathrm{c}} = \widetilde{d}_k(\theta_{k+1}) + \sum_{i=1}^{n_\mathrm{c}} P_{i,k+1} \hat{d}_{k+n_\mathrm{c}-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k+1} z_{k+n_\mathrm{c}-i}. \tag{63}$$

Using (11) and (12), it follows from (63) that

$$\hat{d}_{k+n_\mathrm{c}} = \widetilde{d}_k(\theta_{k+1}) + \hat{d}_{k+n_\mathrm{c}} - D_{\hat{d}z,k+1}(\mathbf{q}) \hat{d}_k + N_{\hat{d}z,k+1}(\mathbf{q}) z_k,$$

which, using (10), can be rewritten as

$$\hat{d}_k = D_{\hat{d}z,k+1}^{-1}(\mathbf{q}) \widetilde{d}_k(\theta_{k+1}) + G_{\hat{d}z,k+1}(\mathbf{q}) z_k. \tag{64}$$

Note that (11), (12), and Proposition 3.4 imply that a state space model corresponding to (64) is given by

$$\hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{G} \widetilde{d}_k(\theta_{k+1}) + \hat{B}_k z_k, \tag{65}$$

$$\hat{d}_k = \hat{C} \hat{x}_k + \hat{D}_k z_k, \tag{66}$$

where $\hat{A}_k$, $\hat{G}$, $\hat{B}_k$, $\hat{C}$, and $\hat{D}_k$ are defined in (46), (47), and (48), and $\hat{x}_0 \triangleq \begin{bmatrix} \hat{d}_{n_\mathrm{c}-1}^\mathrm{T} & \cdots & \hat{d}_0^\mathrm{T} \end{bmatrix}^\mathrm{T}$. Substituting (66) and (62) in (61) yields

$$e_{\mathrm{fc},k+1} = (\overline{A}_k + G_k \hat{D}_k C_k) e_{\mathrm{fc},k} - G_k \hat{C} \hat{x}_k + (B_{\mathrm{a},k} - G_k \hat{D}_k D_{\mathrm{a},k}) \overline{u}_k. \tag{67}$$

Similarly, substituting (62) in (65) yields

$$\hat{x}_{k+1} = \hat{A}_k \hat{x}_k - \hat{B}_k C_k e_{\mathrm{fc},k} + \hat{G} \widetilde{d}_k(\theta_{k+1}) + \hat{B}_k D_{\mathrm{a},k} \overline{u}_k. \tag{68}$$

Define $\widetilde{x}_k \triangleq \begin{bmatrix} \hat{x}_k^\mathrm{T} & e_{\mathrm{fc},k}^\mathrm{T} \end{bmatrix}^\mathrm{T}$. Thus, (57) and (58) follow from (67), (68), and (62). Since $G_{z\widetilde{u},k}$ is the time-domain transfer function of the system represented by (57) and (58), it follows from (56) that

$$z_k = G_{z\overline{u},k}(\mathbf{q}) \overline{u}_k + G_{z\widetilde{d},k}(\mathbf{q}) \widetilde{d}_k(\theta_{k+1}). \tag{69}$$

Finally, substituting (69) in (45) yields (53). ■

Note that the expression for $G_{z\widetilde{u},k}$ is obtained using (51) and (52) in accordance with Definition 3.2 and Proposition

3.3. In order to apply Proposition 3.3, (57) and (58) must be converted to a completely observable state space model. The time-varying Eigensystem Realization Algorithm explained in Section IV of [35] provides a method to reduce any given LTV state space model to a minimal state space model.

In order to analyse the retrospective performance variable decomposition, assume that $R_z = I$, and $\lambda = 1$. Then, it follows from (16) and (53) that

$$J_k(\theta_{k+1}) = \sum_{i=0}^{k} \Big(z_{\mathrm{pp},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{pp},i}(\theta_{i+1}) + z_{\mathrm{mm},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{mm},i}(\theta_{i+1}) + 2 z_{\mathrm{pp},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{mm},i}(\theta_{i+1}) + \theta_{i+1}^{\mathrm{T}} \Phi_i^{\mathrm{T}} R_d \Phi_i \theta_{i+1} \Big) + (\theta_{k+1}-\theta_0)^{\mathrm{T}} R_\theta (\theta_{k+1}-\theta_0). \quad (70)$$

Note that the first two terms in the sum are nonnegative, whereas the third term can have arbitrary sign. This suggests that RLS can minimize $J_k(\theta_{k+1})$ by making the third term negative while the nonnegative terms remain large. In the case where R_θ and R_d is small, using RLS to minimize (70) yields, for $k \ge k_0 \in \mathbb{R}$,

$$z_{\mathrm{rc},k}(\theta_{k+1}) \approx 0, \quad (71)$$

which, using (53), implies that

$$z_{\mathrm{pp},k}(\theta_{k+1}) \approx -z_{\mathrm{mm},k}(\theta_{k+1}). \quad (72)$$

Proposition 4.1: Assume that $\lim_{k\to\infty} \theta_k$ exists and Φ_k is bounded. Then $\lim_{k\to\infty} \widetilde{d}_k(\theta_{k+1}) = 0$.

Proof: Equations (6) and (43) imply that

$$\widetilde{d}_{k-n_{\mathrm{c}}}(\theta_{k+1}) = \hat{d}_k - \Phi_k\theta_{k+1} = \Phi_k(\theta_k - \theta_{k+1}).$$

Defining $\alpha \triangleq \sup_{k\ge 0}\sigma_{\max}(\Phi_k)$, where $\sigma_{\max}$ denotes the maximum singular value, it follows that

$$\|\widetilde{d}_{k-n_{\mathrm{c}}}(\theta_{k+1})\| \le \sigma_{\max}(\Phi_k)\|\theta_k - \theta_{k+1}\| = \alpha\|\theta_k - \theta_{k+1}\|.$$

Hence,

$$\lim_{k\to\infty}\|\widetilde{d}_{k-n_{\mathrm{c}}}(\theta_{k+1})\| \le \alpha \lim_{k\to\infty}\|\theta_k - \theta_{k+1}\| = 0,$$

and thus $\lim_{k\to\infty}\widetilde{d}_{k-n_{\mathrm{c}}}(\theta_{k+1}) = 0$, which in turn implies that $\lim_{k\to\infty}\widetilde{d}_k(\theta_{k+1}) = 0$. ■

V. NUMERICAL EXAMPLE

Consider the state space model given by (1), (2), where, for all $k \ge 0$,

$$A_k \triangleq \begin{bmatrix} 0 & 1 \\ (0.9)^{k+1} & (0.5)^{k+1} \end{bmatrix}, \quad G_k = G \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (73)$$

$$C_k = C \triangleq \begin{bmatrix} 1 & 1.1 \end{bmatrix}, \quad D_{2,k} = D_2 \triangleq 0.01, \quad (74)$$

$u_k = w_k = 0$, v_k is standard Gaussian white noise, and $x_0 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^{\mathrm{T}}$. Let $n_{\mathrm{c}} = 6, n_{\mathrm{f}} = 2, \lambda = 1, R_\theta = 10^{-4} I_{13}, R_d = 10^{-6}, R_z = 1, \widetilde{V} = 10^{-2} I_2$, and let the unknown input be $d_k = 1 + \sin(0.3k)$.

Plots (a) and (b) in Figure 1 show that, after an initial finite number of steps, (71) and (72) hold true. Plot (c) in Figure 1 shows that the difference between z_{rc} and $z_{\mathrm{pp}} + z_{\mathrm{mm}}$ is negligible, and thus confirms (53). The convergence of $\hat{d}$, θ, and $\widetilde{d}$ is depicted in Figure 2. Note that, in these plots, the time step at which the RCIE algorithm is started is assumed as the 0-th step. In order to observe the steady-state behavior of the time-domain transfer functions $G_{z\overline{u}}$ and $G_{z\widetilde{d}}$ after the estimator coefficient θ converges, the magnitude plots of $G_{zd,200}$, $G_{zw,200}$, and $G_{zv,200}$ are shown in Figure 3, where $\begin{bmatrix} G_{zd,200} & G_{zw,200} & G_{zv,200} \end{bmatrix} = G_{z\overline{u},200}$, and the extent to which the frequency response of $G_{z\widetilde{d},200}$ matches with that of $\overline{G}_{\mathrm{f},200}$ is shown in Figure 4.

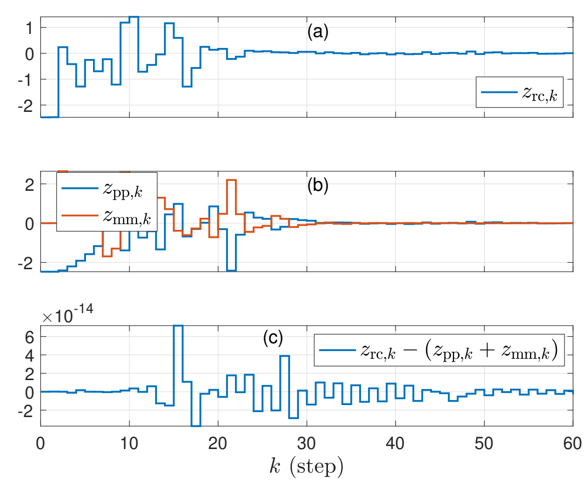

Fig. 1. (a) For all $k \ge 21$, $z_{\mathrm{rc},k} \approx 0$, which confirms (71). (b) For all $k \ge 21$, $z_{\mathrm{pp},k} \approx z_{\mathrm{mm},k}$, which confirms (72). Furthermore, for all $k \ge 35$, $z_{\mathrm{pp},k} \approx z_{\mathrm{mm},k} \approx 0$. (c) For all $k \ge 0$, $|z_{\mathrm{rc},k} - (z_{\mathrm{pp},k} + z_{\mathrm{mm},k})| \le 3 \times 10^{-14}$, which confirms (53).

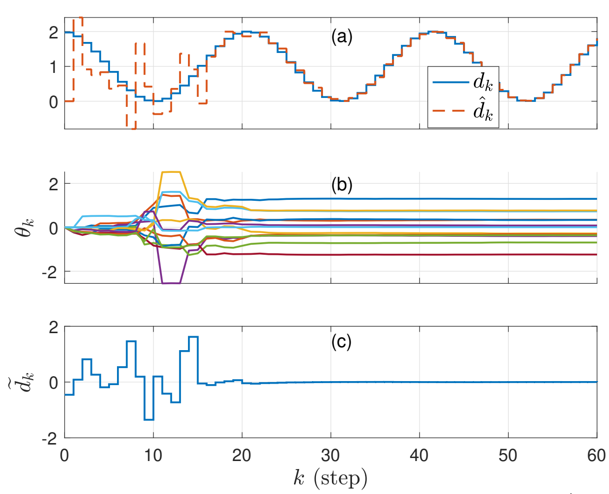

Fig. 2. (a) After the initial transient period of about 25 steps, $\hat{d}$ follows d. (b) The estimator coefficients θ converges after about 25 steps. (c) The virtual external input perturbation $\widetilde{d}$ converges to zero after about 25 steps, in accordance with Proposition 4.1.

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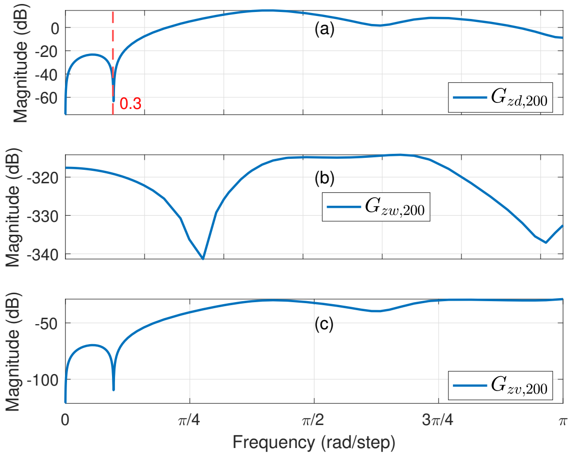

Fig. 3. The magnitudes of $G_{zw,200}$ and $G_{zv,200}$ are close to zero at all frequencies. The magnitude of $G_{zd,200}$ at the frequencies 0 and 0.3 rad/step contained in the spectrum of the unknown input signal d is close to zero. These observations confirm that, for large values of k, $z_{\mathrm{pp},k} \approx 0$.

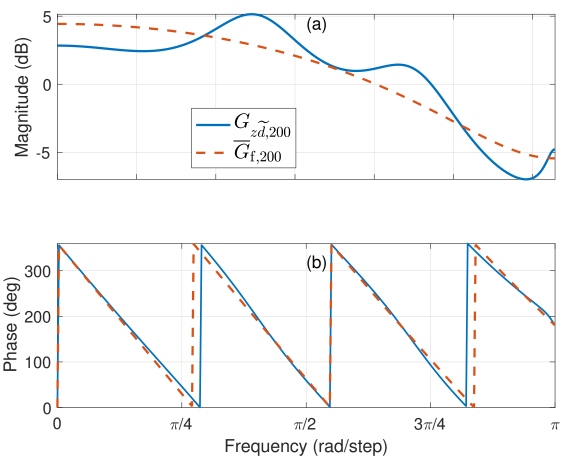

Fig. 4. Comparison of the frequency response of $G_{z\widetilde{d},200}$ with that of $\overline{G}_{\mathrm{f},200}$. The magnitude plots and the phase plots match approximately.

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