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Adaptive Static-Output-Feedback Stabilization with Warm-Start

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***Abstract*— This paper develops an approach to static output feedback under the assumption that a stabilizing static-output-feedback gain is known for an approximate plant model. This approach is motivated by the fact that system identification may be used to obtain an approximate plant model, and offline optimization can be used to obtain a stabilizing static-output-feedback gain. This gain provides the initial guess for the adaptive static-output-feedback control law, which iteratively refines the gain based on the response of the actual system dynamics.**

I. Introduction

A classical approach to linear state-space-based control is to employ a controller of the form $u = Kx$, where the gain matrix K is determined, for example, by pole placement or linear-quadratic regulator. When the state x is not measured and only the output y is available, an observer can be used to obtain an estimate $\hat{x}$ of x, which is used in the output-feedback control law $u = K\hat{x}$. The resulting observer-based compensator is justified by the classical separation principle.

When only the output y is available, it is tempting to employ static output feedback of the form $u = Ky$, which avoids the need for an observer and thus simplifies the feedback control law. Unfortunately, static output feedback is known to be a highly challenging problem, and relevant works include techniques based on quadratic optimization [1], [2] as well pole placement [3]–[8]. The source of the difficulty arises from the fact that the set of stabilizing feedback gains is not necessarily convex [9]. In the single-input, single-output continuous time case, this can be seen clearly from properties of the root locus, which may enter and leave or remain in the open right-half plane.

Despite the challenging nature of the static output feedback problem, the simplicity of this control law in practical implementation motivates the present paper, where the focus is on adaptive static output feedback. Prior work on adaptive static output feedback includes [10], where retrospective cost adaptive control was used to update the static output-feedback gain. The present paper develops an alternative approach to static output feedback under the assumption that a stabilizing static-output-feedback gain is known for an approximate plant model. This approach is motivated by the fact that system identification may be used to obtain an approximate plant model, and offline optimization can be used to obtain a stabilizing static-output-feedback gain. This gain provides the initial guess for the adaptive static-output-feedback control law, which iteratively refines the gain based on the response of the actual system dynamics.

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II. Adaptive Static Output Feedback

Consider the multi-input multi-output (MIMO) input-output discrete time plant

$$y_{k+1} = -\sum_{i=1}^{n} F_i y_{k-i+1} + \sum_{i=1}^{n} G_i u_{k-i+1}, \tag{1}$$

where $k \in \mathbb{N}_0$ is the time step, $n \in \mathbb{N}$ is the plant order, $u_k \in \mathbb{R}^m$ is the control, $y_k \in \mathbb{R}^p$ is the measurement, $F_i \in \mathbb{R}^{p\times p}$ and $G_i \in \mathbb{R}^{p\times m}$ are the plant coefficients, and $y_0, \ldots, y_{1-n}, u_0, \ldots, u_{1-n}$ are the initial conditions. We assume there exists a static output feedback (SOF) gain $K \in \mathbb{R}^{m\times p}$ such that the static output feedback controller

$$u_k = K y_k \tag{2}$$

stabilizes (1). We say, in short, that K stabilizes (1) if there exists a state-space realization

$$x_{k+1} = Ax_k + Bu_k \tag{3}$$
$$y_k = Cx_k \tag{4}$$

of (1) such that all of the eigenvalues of $A + BKC$ are in the open unit disc. Conditions under which (1) is stabilizable are discussed in [9], [11].

Next, we consider a MIMO input-output model of the plant (1) of the form

$$\bar{y}_{k+1} = -\sum_{i=1}^{\bar{n}} \bar{F}_i \bar{y}_{k-i+1} + \sum_{i=1}^{\bar{n}} \bar{G}_i \bar{u}_{k-i+1}, \tag{5}$$

where $\bar{n} \in \mathbb{N}$ is the model order, $\bar{F}_i \in \mathbb{R}^{p\times p}$ and $\bar{G}_i \in \mathbb{R}^{p\times m}$ are the model coefficients, and $\bar{y}_0, \ldots, \bar{y}_{1-n}, \bar{u}_0, \ldots, \bar{u}_{1-n}$ are the model initial conditions. We assume that a matrix $\bar{K} \in \mathbb{R}^{m\times p}$ that stabilizes (5) exists and is known.

The problem we consider is, with knowledge of the model (5) and stabilizing gain $\bar{K}$, to design an adaptive static output feedback gain $\theta_k \in \mathbb{R}^{m\times p}$ such that the output feedback controller

$$u_k = \theta_k y_k, \tag{6}$$

stabilizes (1). Our proposed algorithm is given in the following section III.

III. Adaptive Algorithm

Define $\theta_0 \triangleq \bar{K}$, let the forgetting factor $\lambda \in \mathbb{R}$ be positive, and let $P_0 \in \mathbb{R}^{mp\times mp}$ be (symmetric) positive definite. For all $k \geq 0$, denote the minimizer of the function

$$J_k(\theta) \triangleq \sum_{j=0}^{k} \lambda^{k-j} \left\| \overline{\Big(\sum_{i=1}^{\bar{n}} G_i u_{j-i+1}} - \sum_{i=1}^{\bar{n}} \bar{G}_i \bar{K} y_{j-i+1}\Big) - \Big(\sum_{i=1}^{\bar{n}} \bar{G}_i u_{j-i+1} - \sum_{i=1}^{\bar{n}} \bar{G}_i \theta y_{j-i+1}\Big) \right\|^2 + (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0)^{\mathrm{T}} P_0^{-1} (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0) \tag{7}$$

by $\theta_{k+1} \triangleq \mathrm{argmin}_{\theta \in \mathbb{R}^{m \times p}} J_k(\theta)$, where

$$\overline{\sum_{i=1}^{\bar{n}} G_i u_{k-i+1}} \triangleq y_{k+1} + \sum_{i=1}^{\bar{n}} \bar{F}_i y_{k-i+1}, \tag{8}$$

vec is the column-stacking operator, and $\|\cdot\|$ is the Euclidean norm. In (7), the forgetting factor $0 < \lambda < 1$ provides higher weighting to more recent data, whereas $\lambda > 1$ provides lower weighting to more recent data. Moreover, the matrix P_0 provides regularization by weighting θ relative to the initial gain θ_0 and ensures that J_k has a global minimizer.

To obtain a recursive update for θ_k, we rewrite (7) as

$$J_k(\theta) = \sum_{j=0}^{k} \lambda^{k-j} (Y_j - \Phi_j \mathrm{vec}\,\theta)^{\mathrm{T}} (Y_j - \Phi_j \mathrm{vec}\,\theta) + (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0)^{\mathrm{T}} P_0^{-1} (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0), \tag{9}$$

where

$$\Phi_k \triangleq -\sum_{i=1}^{\bar{n}} \bar{G}_i (y_{k-i+1}^{\mathrm{T}} \otimes I_m), \tag{10}$$

$$Y_k \triangleq \overline{\sum_{i=1}^{\bar{n}} G_i u_{k-i+1}} - \sum_{i=1}^{\bar{n}} \bar{G}_i \bar{K} y_{k-i+1} - \sum_{i=1}^{\bar{n}} \bar{G}_i u_{k-i+1}, \tag{11}$$

and $\otimes$ denotes the Kronecker product. Then, for all $k \geq 0$, it follows from recursive least squares (RLS) [12] that θ_{k+1} is given by

$$P_{k+1} = \frac{1}{\lambda} P_k - \frac{1}{\lambda} P_k \Phi_k^{\mathrm{T}} (I_p + \Phi_k P_k \Phi_k^{\mathrm{T}})^{-1} \Phi_k P_k, \tag{12}$$

$$\mathrm{vec}\,\theta_{k+1} = \mathrm{vec}\,\theta_k + P_{k+1} \Phi_k^{\mathrm{T}} (Y_k - \Phi_k \mathrm{vec}\,\theta_k). \tag{13}$$

It can be seen that, if the coefficients and initial conditions of plant and model match, that is, for $i = 1, \ldots, n$, $\bar{F}_i = F_i$ and $\bar{G}_i = G_i$ and, for all $k \leq 0$, $\bar{y}_k = y_k$ and $\bar{u}_k = u_k$, then, for all $k \geq 0$, the cost function (7) simplifies to

$$J_k(\theta) = \sum_{j=0}^{k} \lambda^{k-j} \left\| \Big(\sum_{i=1}^{\bar{n}} \bar{G}_i \theta y_{j-i+1} - \sum_{i=1}^{\bar{n}} \bar{G}_i \bar{K} y_{j-i+1}\Big) \right\|^2 + (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0)^{\mathrm{T}} P_0^{-1} (\mathrm{vec}\,\theta - \mathrm{vec}\,\theta_0), \tag{14}$$

and, for all $k \geq 0$, $\theta_k = \theta_0$.

As data are collected online from the plant (1), it may be possible to update the model (5) and redesign a stabilizing SOF controller for the updated model. Designing a stabilizing SOF controller is NP-hard [13], however, and thus may not be feasible for online computation.

In contrast, the least squares estimate, θ_k, is computed recursively in (12) and (13) and requires only moderate computations. Furthermore, design of the initial feedback matrix $\bar{K}$ can be done offline prior to operation.

IV. Application to First-Order, SISO plants

Consider the first-order, SISO plant

$$y_{k+1} = -a y_k + b u_k, \tag{15}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R} \backslash \{0\}$, and, for all $k \geq 0$, $y_k, u_k \in \mathbb{R}$. Using the controller (2), the plant (15) can be written as

$$y_{k+1} = (-a + bK) y_k, \tag{16}$$

where $K \in \mathbb{R}$. Note that K stabilizes (15) if and only if $-a + bK \in (-1, 1)$. Assume the model of (15) is given by

$$\bar{y}_{k+1} = -\bar{a} \bar{y}_k + \bar{b} \bar{u}_k, \tag{17}$$

where $\bar{a} \in \mathbb{R}$ and $\bar{b} \in \mathbb{R} \backslash \{0\}$. We choose the gain $\bar{K} = \bar{a}/\bar{b}$ for model (17), which sets $-\bar{a} + \bar{b}\bar{K} = 0$. We also select $\lambda = 1$ for no forgetting.

Next, the cost function (7) can be written

$$J_k(\theta) = \sum_{j=0}^{k} (y_{j+1} + \bar{a} y_j - \bar{b}\bar{K} y_j - \bar{b} u_j + \bar{b}\theta y_j)^2 + P_0 (\theta - \theta_0)^2, \tag{18}$$

where $\theta_0 = \bar{a}/\bar{b}$ and $P_0 > 0$. The adaptive SOF controller is then given by (6), where, for all $k \geq 0$, the recursive update equations (12) and (13) for the adaptive SOF gain $\theta_k \in \mathbb{R}$ can be written as

$$P_{k+1} = \frac{P_k}{1 + \bar{b}^2 y_k^2 P_k}, \tag{19}$$

$$\begin{aligned} \theta_{k+1} &= \theta_k - P_{k+1} \bar{b} y_k (y_{k+1} + \bar{a} y_k - \bar{b}\bar{K} y_k - \bar{b} u_k + \bar{b}\theta_k y_k) \\ &= \theta_k - P_{k+1} \bar{b} y_k^2 \left[(-a + b\theta_k) - (-\bar{a} + \bar{b}\bar{K})\right] \\ &= \theta_k - P_{k+1} \bar{b} y_k^2 (-a + b\theta_k). \end{aligned} \tag{20}$$

Finally, the closed-loop plant dynamics can be written as

$$y_{k+1} = (-a + b\theta_k) y_k. \tag{21}$$

Next, for all $k \geq 0$, define

$$R_k \triangleq P_{k+1} \bar{b}^2 y_k^2, \tag{22}$$

$$\psi_k \triangleq -a + b\theta_k. \tag{23}$$

Then, for all $k \geq 0$, it follows from (19) and (20) that

$$R_{k+1} = \frac{\psi_k^2 R_k}{1 + \psi_k^2 R_k}, \tag{24}$$

$$\psi_{k+1} = (1 - \frac{b}{\bar{b}} R_k) \psi_k. \tag{25}$$

Proposition 1. *If $y_0 \neq 0$, then $0 < R_0 < 1$ and, for all $k \geq 1$, $0 \leq R_k < 1$.*

Proof. Note that $R_0 = P_1 \bar{b}^2 y_0^2 = \frac{P_0 \bar{b}^2 y_0^2}{1 + P_0 \bar{b}^2 y_0^2} < \frac{1 + P_0 \bar{b}^2 y_0^2}{1 + P_0 \bar{b}^2 y_0^2} = 1$. Furthermore, $P_0 > 0$ and $y_0 \neq 0$ imply that $P_0 \bar{b}^2 y_0^2 > 0$, and thus $R_0 = \frac{P_0 \bar{b}^2 y_0^2}{1 + P_0 \bar{b}^2 y_0^2} > 0$. Next, by (24), $0 \leq R_k < 1$ implies that $0 \leq R_{k+1} < 1$. □

Lemma 1. *Let* $\psi_k^2 > 1$ *and* $0 < R_k < 1$.

i) If $0 < R_k < 1 - \frac{1}{\psi_k^2}$, *then* $R_k < R_{k+1} < 1 - \frac{1}{\psi_k^2}$.

ii) If $R_k = 1 - \frac{1}{\psi_k^2}$, *then* $R_{k+1} = 1 - \frac{1}{\psi_k^2}$.

iii) If $1 - \frac{1}{\psi_k^2} < R_k$, *then* $1 - \frac{1}{\psi_k^2} < R_{k+1} < R_k$.

Proof. ii) follows from substituting $R_k = 1 - \frac{1}{\psi_k^2}$ into (24). *i)* and *iii)* can be easily verified as properties of the difference equation (24). $\square$

Theorem 2. *If* $y_0 \neq 0$ *and* $0 < b/\bar{b} < 2$, *then* $\lim_{k\to\infty}(-a + b\theta_k)$ *exists and*

$$\lim_{k\to\infty}(-a + b\theta_k) \in (-1, 1). \tag{26}$$

Proof. First, we show that $|\psi_k| = |-a + b\theta_k|$ is decreasing. Note that, for all $k \geq 0$, since $0 < b/\bar{b} < 2$ and $0 \leq R_k \leq 1$ from Proposition 1, it follows that $0 \leq b/\bar{b} R_k \leq 2$. It then follows that $-1 \leq 1 - b/\bar{b} R_k \leq 1$. Applying this inequality to (25), it follows that $|\psi_{k+1}| \leq |\psi_k|$. Since $|\psi_k|$ is decreasing and bounded below by 0, it follows that $\lim_{k\to\infty} |\psi_k|$ exists.

Next, suppose, for contradiction, that $\lim_{k\to\infty} |\psi_k| > 1$. Then there exists $N \in \mathbb{N}$ such that, for all $k \geq N$, $\psi_k \neq 0$. Hence, it follows from (25) that, for all $k \geq N$, $\frac{|\psi_{k+1}|}{|\psi_k|} = \left|1 - \frac{b}{\bar{b}} R_k\right|$. Then, taking the limit as $k \to \infty$, we have $\lim_{k\to\infty}\left|1 - \frac{b}{\bar{b}} R_k\right| = 1$. Thus, for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that, for all $k > M$, either $|b/\bar{b} R_k| < \varepsilon$ or $|b/\bar{b} R_k - 2| < \varepsilon$.

However, since $0 < b/\bar{b} < 2$ by assumption, there exists $\alpha > 0$ such that $0 < b/\bar{b} < 2 - \alpha$. Then, since for all $k \geq 0$, $0 < R_k < 1$ by Proposition 1, it follows that, for all $k \geq 0$, $0 \leq \frac{b}{\bar{b}} R_k \leq 2 - \alpha$, which implies that $\left|\frac{b}{\bar{b}} R_k - 2\right| \geq \alpha$.

Hence, if $\varepsilon < \alpha$, then there is no $M \in \mathbb{N}$ such that, for all $k \geq M$, $|b/\bar{b} R_k - 2| < \varepsilon$. Therefore, for all ε such that $0 < \varepsilon < \alpha$, there exists M such that, for all $k > M$, $|b/\bar{b} R_k| < \varepsilon$. This implies that $\lim_{k\to\infty} R_k = 0$.

Next, since $|\psi_k|$ is decreasing and $\lim_{k\to\infty} |\psi_k| > 1$ is assumed for contradiction, it follows that, for all $k \geq 0$, $|\psi_k| > 1 + \delta$ for some $\delta > 0$. This implies that, for all $k \geq 0$, $1 - \frac{1}{\psi_k^2} > 0$. Moreover, since $R_0 > 0$ from Proposition 1 and since, for all $k \geq 0$, $|\psi_k| > 1$, it follows from (24) that, for all $k \geq 0$, $R_k > 0$. Hence, $\lim_{k\to\infty} R_k = 0$ implies that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $0 < R_k < \varepsilon$.

However, it follows from Lemma 1 that if $0 < R_k < 1 - \frac{1}{\psi_k^2}$, $R_{k+1} > R_k$. and if $R_k \geq 1 - \frac{1}{\psi_k^2}$, $R_{k+1} \geq 1 - \frac{1}{\psi_k^2}$. Since, for all $k \geq 0$, $|\psi_k| > 1 + \delta$ for some $\delta > 0$, it follows that Lemma 1 contradicts $\lim_{k\to\infty} R_k = 0$. Hence, we conclude that $\lim_{k\to\infty} |\psi_k| \leq 1$.

Next, suppose, for contradiction, that $\lim_{k\to\infty} |\psi_k| = 1$. Since $|\psi_k|$ is decreasing, it follows for Lemma 1 that, for all $k \geq 0$, $\min\{R_0, 1 - \frac{1}{\psi_k^2}\} \leq R_k \leq \max\{R_0, 1 - \frac{1}{\psi_0^2}\}$. Moreover, it follows by assumption that $\lim_{k\to\infty} 1 - \frac{1}{\psi_k^2} = 0$. Hence, for sufficiently large N, $|1 - \frac{b}{\bar{b}} R_N| < 1 - (1 - \frac{1}{\psi_N^2}) = \frac{1}{\psi_N^2}$. Applying this inequality to (25), it follows that $|\psi_{N+1}| = |1 - \frac{b}{\bar{b}} R_N||\psi_N| < |\frac{1}{\psi_N^2}||\psi_N| = \frac{1}{|\psi_N|}$. Since, for all $k \geq 0$, $|\psi_k|$ is decreasing and $\lim_{k\to\infty} |\psi_k| = 1$ is assumed, it follows that, for all $k \geq 0$, $|\psi_k| > 1$. Hence, $|\psi_N| > 1$ which implies $|\psi_{N+1}| < 1$, a contradiction.

Hence, we conclude $\lim_{k\to\infty} |\psi_k| < 1$. It follows that there exists $N \in \mathbb{N}$ such that, for all $k \geq N$, $0 \leq \psi_k^2 < 1$. It then follows from Proposition 1 that $0 \leq R_N \psi_N^2 < 1$. Furthermore, since $f(x) = \frac{x}{1+x}$ is increasing for $x \geq 0$ and $f(1) = \frac{1}{2}$, it follows that $0 \leq R_{N+1} = \frac{\psi_N^2 R_N}{1 + \psi_N^2 R_N} < \frac{1}{2}$.

Finally, for all $k \geq N + 1$, since $1 + \psi_k^2 R_k \geq 1$ and $0 \leq \psi_k^2 < 1$, it follows that $R_{k+1} = \frac{\psi_k^2 R_k}{1 + \psi_k^2 R_k} \leq \psi_k^2 R_k < R_k$. Hence, for all $k \geq N + 1$, $0 \leq R_k < \frac{1}{2}$. Next, since $0 < \frac{b}{\bar{b}} < 2$, it follows that, for all $k \geq N + 1$, $0 \leq 1 - \frac{b}{\bar{b}} R_k < 1$. Applying this inequality to (25), it follows that, for all $k \geq N + 1$, the sign of ψ_k is the same. This property and $\lim_{k\to\infty} |\psi_k| < 1$ imply that $\lim_{k\to\infty} \psi_k$ exists and satisfies $\lim_{k\to\infty} \psi_k \in (-1, 1)$. $\square$

Theorem 2 shows that, for a first-order single-input single-output (SISO) plant, knowledge of the DC gain within -6 dB and ∞ dB, that is, $0 < b/\bar{b} < 2$, is sufficient for global convergence to a stabilizing SOF gain.

V. Numerical Examples

Next, we study the ability of the proposed adaptive SOF algorithm to stabilize higher order linear SISO and MIMO plants through three numerical examples.

A. Example 1: Second Order SISO

Consider a 2[nd] order discrete-time SISO plant

$$y_{k+1} = 2y_k + 0.25y_{k-1} + 2.25u_k + 1.25u_{k-1} + w_k, \tag{27}$$

where, for all $k \geq 0$, $y_k \in \mathbb{R}$, $u_k \in \mathbb{R}$, and $w_k \sim \mathcal{N}(0, 1)$. In the case without noise w_k, (27) has the transfer function

$$H(z) = \frac{2.25\,z + 1.25}{z^2 - 2\,z - 0.25}. \tag{28}$$

To obtain a model for (27), we first simulate a step response of the (27) with the initial conditions $y_0 = y_{-1} = 0$ and $u_0 = u_{-1} = 1$ and, for all $k > 0$, $u_k = 1$. The step response is simulated for $0 \leq k \leq 20$ and MATLAB's `tfest` is used to fit a 2[nd] order model from the resulting data, $\{u_k\}_{k=0}^{20}$ and $\{y_k\}_{k=0}^{20}$. The model is given by the discrete-time transfer function

$$\bar{H}(z) = \frac{3.901\,z - 4.977}{z^2 - 1.277\,z + 0.2773}. \tag{29}$$

The combination of $H(z)$ being open-loop unstable, measurement noise, and short simulation time result in a very poor model $\bar{H}(z)$ of plant $H(z)$. To highlight the differences between the poles and zeros of the plant and model, we can rewrite (28) and (29), respectively, as

$$H(z) = \frac{2.25(z + 0.5556)}{(z - 2.118)(z + 0.118)}, \tag{30}$$

$$\bar{H}(z) = \frac{3.901(z + 1.276)}{(z - 0.9996)(z - 0.2774)}. \tag{31}$$

The root loci of $H(z)$ and $\bar{H}(z)$ are shown in Figure 1 where the closed-loop poles are, respectively, the roots of

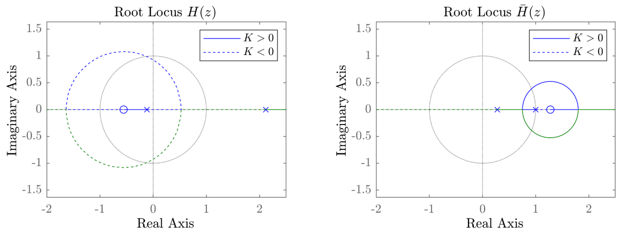

Fig. 1: Root loci of $H(z)$ given by (27) and $\bar{H}(z)$ given by (29).

$1 - KH(z) = 0$ and $1 - K\bar{H}(z) = 0$ to match the sign convention of (2). Plant $H(z)$ is closed-loop stable with SOF gain $K \in (-1, -0.3571)$ and model $\bar{H}(z)$ is closed-loop stable with SOF gain $K \in (-0.0003, 0.1452)$. Note that there is no overlap in the sets of stabilizing SOF gains for $H(z)$ and $\bar{H}(z)$.

Stabilizing model SOF gain $\bar{K}$ is chosen to minimize the closed-loop model's largest absolute value of its poles. Using MATLAB's `fminunc`, this is calculated to be $\bar{K} = 0.0575$. We use tuning parameters $P_0 = 0.1$ and $\lambda = 1$ and recall that $\theta_0 \triangleq \bar{K}$. We simulate the plant (27) with plant initial conditions $y_0 = y_{-1} = 1$ and $u_0 = \theta_0 y_0$, $u_{-1} = \theta_0 y_{-1}$, model initial conditions $\bar{y}_0 = \bar{y}_{-1} = 0$ and $u_0 = u_{-1} = 0$, and, for all $k \geq 0$, $w_k \sim \mathcal{N}(0, 1)$.

The simulation results are shown in Figure 2. These results show quick adaptation to a stabilizing SOF gain despite measurement noise and significant differences between plant and model.

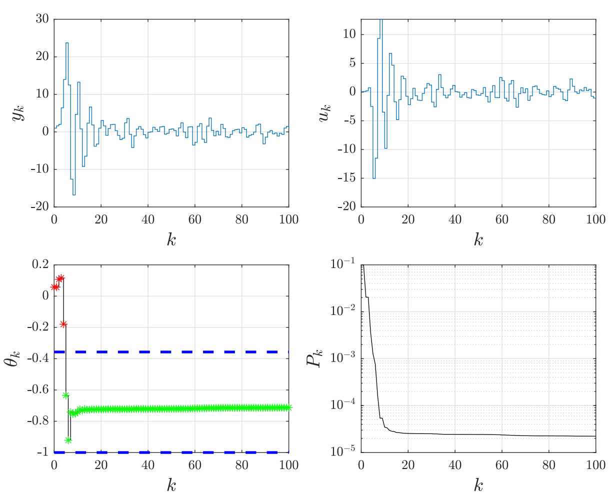

Fig. 2: Example 1 simulation results: Plots show measurement y_k, control u_k, adaptive SOF gain θ_k, and covariance P_k of θ_k. In the bottom left plot, blue dashed indicates the boundary between stabilizing and non-stabilizing SOF gains, red star indicates θ_k is not stabilizing, and green star indicates θ_k is stabilizing.

B. Example 2: Second to Fifth Order SISO

In this example, we test our adaptive SOF algorithm on a large collection of SISO plants between 2nd and 5th order, systematically varying the difference between plant and model.

Consider an $n^{\rm th}$ order SISO plant of the form

$$y_{k+1} = -\sum_{i=1}^{n} \frac{a_i}{a_0} y_{k-i+1} + \sum_{i=1}^{n} \frac{b_i}{a_0} u_{k-i+1} + w_k, \qquad (32)$$

where $a_0 \in (-1, 1)\backslash\{0\}$, for all $1 \leq i \leq n$, $a_i, b_i \in (-1, 1)$, for all $k \geq 0$, $y_k, u_k \in \mathbb{R}$, and noise $w_k \sim \mathcal{N}(0, 1)$. In the absence of noise w_k, the plant (32) can be represented by the discrete-time transfer function

$$H(z) = \frac{b_1 z^{-1} + \cdots + b_{n-1} z^{-(n-1)} + b_n z^{-n}}{a_0 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n}}. \qquad (33)$$

The transfer function (33) can be rewritten as

$$H(z) = k \frac{(z - z_1) \cdots (z - z_{n-1})}{(z - p_1) \cdots (z - p_{n-1})(z - p_n)}, \qquad (34)$$

where $z_1, \cdots, z_{n-1} \in \mathbb{C}$ are the $n-1$ zeros of the plant, $p_1, \cdots, p_n \in \mathbb{C}$ are the n poles of the plant, and $k \in \mathbb{R}$ is the leading coefficient of the plant.

We consider a model with the transfer function

$$\bar{H}_\kappa(z) = \kappa k \frac{(z - z_1) \cdots (z - z_{n-1})}{(z - p_1) \cdots (z - p_{n-1})(z - p_n)}, \qquad (35)$$

for $\kappa \in \mathbb{R}$. If (35) is SOF stabilizable, the stabilizing gain $\bar{K}$ for model (35) is selected to minimize the closed-loop model's largest absolute value of its poles. Note that for a SISO plant, the set of stabilizing SOF gains is the union of finitely many open intervals and $\bar{K}$ was computed by using MATLAB's `fminunc` to minimize the closed-loop model's largest absolute value of its poles in each open interval.

To generate a collection of plants, for all $1 < i < n$, we sample a_i, b_i from the uniform distribution on $(-1, 1)$ and also sample a_0 from the uniform distribution on $(-1, 1)$. Note that there is probability 0 that $a_0 = 0$. We select the first 100 cases in which all of the following:

i) plant (32) is SOF stabilizable,
ii) model (35) is SOF stabilizable,
iii) gain $\bar{K}$ stabilizes plant (32).

Next, we also select the first 100 cases in which *i)*, *ii)*, and gain $\bar{K}$ does not stabilize plant (32). We perform this process for plant order $n = 2, 3, 4, 5$.

In each simulation, the plant initial conditions are $y_i = 1$ and $u_i = \bar{K} y_i$ for $i = 0, -1, \ldots, 1-n$, and the model initial conditions are $y_i = 0$ and $u_i = 0$ for $i = 0, -1, \ldots, 1-n$. We select tuning parameters $P_0 = 0.1$ and $\lambda = 1$ and recall that $\theta_0 \triangleq \bar{K}$.

We simulate each adaptive controller for $N = 1000$ steps. We say that a plant is "stabilized" if for all $900 \leq k \leq 1000$, θ_k stabilizes plant (32) under SOF. The results of these simulations are summarized in Table I for $\kappa = 0.5$ and in Table II for $\kappa = 2$.

TABLE I: Percentage of plants stabilized using model (35) with $\kappa = 0.5$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	97%	97%
θ_0 not stabilizing	99%	100%	99%	97%

TABLE II: Percentage of plants stabilized using model (35) with $\kappa = 2.0$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	100%	99%
θ_0 not stabilizing	100%	100%	99%	100%

Next, the same process is repeated with the model

$$\bar{H}_\alpha(z) = k\frac{(z-z_1)\cdots(z-z_{n-1})}{(z-\alpha p_1)\cdots(z-\alpha p_{n-1})(z-\alpha p_n)}, \qquad (36)$$

obtained by scaling the plant poles by $\alpha \in \mathbb{R}$. The result are summarized in Table III for $\alpha = 0.8$ and in Table IV for $\alpha = 1.2$.

TABLE III: Percentage of plants stabilized using model (36) with $\alpha = 0.8$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	99%	100%
θ_0 not stabilizing	96%	90%	84%	86%

TABLE IV: Percentage of plants stabilized using model (36) with $\alpha = 1.2$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	100%	100%
θ_0 not stabilizing	94%	96%	87%	92%

Lastly, the process is repeated with the model

$$\bar{H}_\beta(z) = k\frac{(z-\beta z_1)\cdots(z-\beta z_{n-1})}{(z-p_1)\cdots(z-p_{n-1})(z-p_n)}, \qquad (37)$$

obtained by scaling the plant zeros by $\beta \in \mathbb{R}$. The result are summarized in Table V for $\beta = 0.8$ and in Table VI for $\beta = 1.2$.

TABLE V: Percentage of plants stabilized using model (37) with $\beta = 0.8$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	99%	99%
θ_0 not stabilizing	97%	98%	88%	85%

TABLE VI: Percentage of plants stabilized using model (37) with $\beta = 1.2$

	2nd Order	3rd Order	4th Order	5th Order
θ_0 stabilizing	100%	100%	97%	97%
θ_0 not stabilizing	94%	92%	89%	86%

These results show that with a model generated from DC gain mismatch, pole scaling, and zero scaling, the adaptive SOF algorithm is able to stabilize nearly all test cases with θ_0 stabilizing and a majority of cases with θ_0 not stabilizing.

1) Considerations of Noise and $\lambda \neq 1$: In some simulations, additive white noise $w_k \sim \mathcal{N}(0,1)$ had the effect of θ_k slowly drifting after settling. An potential solution to drifting is to select $\lambda > 1$. Selecting $\lambda = 1.001$ and extending the simulation time to $N = 5000$ results in 100% of plants summarized in Tables I and II having stabilizing θ_k for $4000 \leq k \leq 5000$ in the presence of additive white noise $w_k \sim \mathcal{N}(0,1)$.

On the other hand, selecting $\lambda \geq 1$ implies that for all $k \geq 0$, $P_{k+1} \preceq P_k$ [14]. This results in slowing or halting of adaptation which is not desirable if, for example, the plant dynamics are changing. In such situations, either $\lambda < 1$ or variable-rate forgetting [15] can be used to allow $P_{k+1} \succeq P_k$, and therefore, continued adaptation.

C. Example 3: Third Order MIMO

We consider a MIMO plant from [16] and use the static output feedback H_∞ controller designed in [16] for the model warm-start in our adaptive controller. We then study the robustness of the SOF H_∞ controller versus our adaptive controller to differences between model and plant.

Consider the MIMO state-space plant

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k + B_u u_k + B_w w_k, \\ y_k &= Cx_k, \end{aligned} \qquad (38)$$

where

$$A(\alpha) \triangleq \begin{bmatrix} \alpha & 0.3 & 2 \\ 1 & 0 & 1 \\ 0.3 & 0.6 & -0.6 \end{bmatrix}, \; B_u \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B_w \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$C \triangleq \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

$\alpha \in \mathbb{R}$, and, for all $k \geq 0$, $x_k \in \mathbb{R}^3$, $y_k \in \mathbb{R}$, $u_k \in \mathbb{R}^2$, and $w_k \sim \mathcal{N}(0, \sigma^2)$.

Next, the model we will be using is

$$\begin{aligned} x_{k+1} &= \bar{A}(\bar{\alpha})x_k + B_u u_k + B_w w_k, \\ y_k &= Cx_k, \end{aligned} \qquad (39)$$

where

$$\bar{A}(\bar{\alpha}) \triangleq \begin{bmatrix} \bar{\alpha} & 0.3 & 2 \\ 1 & 0 & 1 \\ 0.3 & 0.6 & -0.6 \end{bmatrix}, \qquad (40)$$

and $\bar{\alpha} \in \mathbb{R}$. The model (39) can be rewritten in input-output form as

$$\bar{y}_{k+1} = -\sum_{i=1}^{3} \bar{F}_i \bar{y}_{k-i+1} + \sum_{i=1}^{3} \bar{G}_i \bar{u}_{k-i+1}, \qquad (41)$$

where for $i = 1, 2, 3$, $\bar{F}_i \in \mathbb{R}$ and $\bar{G}_i \in \mathbb{R}^{1\times 2}$ are found from

$$C(zI - \bar{A}(\bar{\alpha}))^{-1}B_u = \frac{\bar{G}_1 z^{-1} + \bar{G}_2 z^{-2} + \bar{G}_3 z^{-3}}{1 + \bar{F}_1 z^{-1} + \bar{F}_2 z^{-2} + \bar{F}_3 z^{-3}}. \qquad (42)$$

We consider $\bar{\alpha} = 1.9$, 2.7, 2.8, or 2.9 and use the SOF H_∞ controller designed in [16] for $\bar{K}$, as shown in the first two columns of Table VII. Note that if there is no noise, i.e. $\sigma^2 = 0$, $\bar{K}$ stabilizes (38) whenever the eigenvalues of $A(\alpha)+B_u KC$ are all in the open unit disc. The set of α such that, without noise, $\bar{K}$ stabilizes (38) is found numerically and presented in column 3 of Table VII.

TABLE VII: $\bar{K}$ is the SOF H_∞ controller for model (39), as designed in [16]. Column 3 gives the set of α such that $\bar{K}$ stabilizes plant (38).

$\bar{\alpha}$	$\bar{K}$	α interval s.t. $\bar{K}$ stabilizing
1.9	$\begin{bmatrix} -0.6637 & -0.4965 \end{bmatrix}^{\mathrm{T}}$	$(0.6512, 2.1980)$
2.7	$\begin{bmatrix} -0.9353 & -0.4686 \end{bmatrix}^{\mathrm{T}}$	$(2.0861, 2.8138)$
2.8	$\begin{bmatrix} -0.9654 & -0.4402 \end{bmatrix}^{\mathrm{T}}$	$(2.2412, 2.8939)$
2.9	$\begin{bmatrix} -0.9932 & -0.4089 \end{bmatrix}^{\mathrm{T}}$	$(2.3881, 2.9727)$

Additionally, let $\rho(\cdot)$ denote the spectral radius of a square matrix. Figure 3 shows, shaded in blue, $\rho(A(\alpha) + B_u KC)$ for all α such that $\rho(A(\alpha) + B_u \bar{K}C) < 1$.

To compare the robustness to difference between $\bar{\alpha}$ and α of the fixed gain SOF controller $u_k = \bar{K}y_k$ versus our adaptive SOF controller $u_k = \theta_k y_k$, we test the adaptive controller on plant (38) for $-3 \le \alpha \le 6$ in increments of 0.02 with $\sigma^2 = 0$. We select tuning parameters $P_0 = 1$, $\lambda = 1$, plant initial condition $x_0 = [1\ 1\ 1]^{\mathrm{T}}$, and model initial conditions $\bar{y}_i = 0$, $\bar{u}_i = [0\ 0]^{\mathrm{T}}$ for $i = 0, -1, -2$.

For each α, we simulate for $0 \le k \le 5000$ and consider the plant (38) stabilized if, for all $4000 \le k \le 5000$, θ_k stabilizes the plant. If $\rho(A(\alpha) + B_u\theta_k C)$ has diverged, we plot a red line downward in Figure 3. Otherwise, we plot an upward line with height $\rho(A(\alpha) + B_u\theta_{5000}C)$ in green if the plant is stabilized and in red if not. Figure 3 shows, for each $\bar{\alpha}$ chosen, we greatly increase the set of α over which the plant (38) is stabilized.

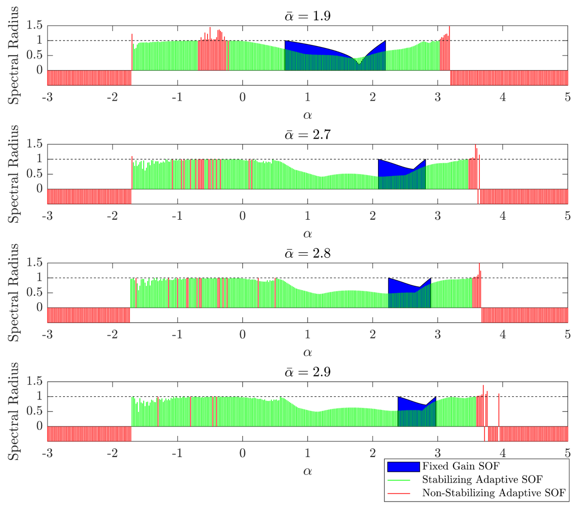

Fig. 3: Numerical stability analysis of non-adaptive and adaptive SOF controllers on the plant (38) with different values of $\bar{\alpha}$ and α. We plot $\rho(A(\alpha) + B_u\bar{K}C)$ in blue solid. Next, for $-3 \le \alpha \le 5$ in increments of 0.02, if $\rho(A(\alpha) + B_u\theta_k C)$ has diverged, we plot a red line downward. Otherwise, we plot a line upward with height $\rho(A(\alpha) + B_u\theta_{5000}C)$ in green or red, where green indicates that, for all $4000 \le k \le 5000$, $\rho(A(\alpha) + B_u\theta_k C) < 1$ and red indicates otherwise.

To better illustrate the adaptive controller, consider the case $\alpha = 2.5$, $\bar{\alpha} = 1.9$, and $\bar{K} = [-0.6637\ -0.4965]^{\mathrm{T}}$, with noise $w_k \sim \mathcal{N}(0, 1)$. The matrix $A(\alpha) + B_u\bar{K}C$ has spectral radius 1.4016 and thus $\bar{K}$ does not stabilize the plant (38). Next, we simulate (38) with the adaptive SOF controller with the same parameters and initial conditions as used previously. Figure 4 shows the response for $0 \le k \le 100$. The matrix $A(\alpha) + B_u\theta_{100}C$ has spectral radius 0.9374 and thus θ_{100} stabilizes the plant (38).

VI. Conclusions and Future Work

Motivated by the longstanding problem of stabilizing linear time-invariant plants by means of static output feedback, this paper developed a warm-start adaptive static-output-feedback control algorithm. Convergence of this algorithm was proved for SISO first-order plants and demonstrated numerically for higher order SISO and MIMO plants. Future research will focus on extending convergence proofs to higher order SISO plants and to MIMO plants.

VII. Acknowledgments

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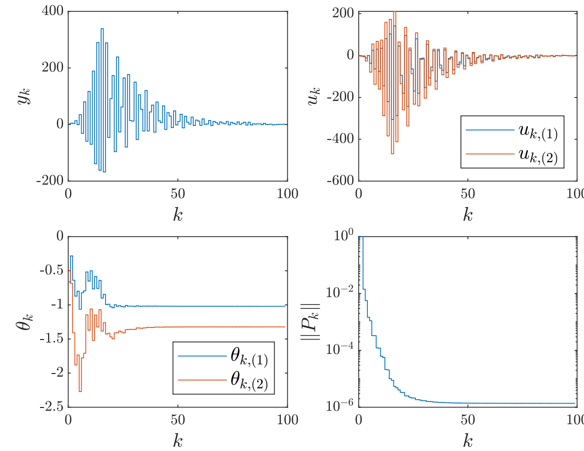

Fig. 4: Simulation of adaptive SOF algorithm on plant (38) with model (39) where $\alpha = 2.5$, $\sigma^2 = 1$, and $\bar{\alpha} = 1.9$. The tuning parameters selected are $P_0 = 1$, and $\lambda = 1$, the plant initial condition is $x_0 = [1\ 1\ 1]^{\mathrm{T}}$, and the model initial conditions are $\bar{y}_i = 0$, $\bar{u}_i = [0\ 0]^{\mathrm{T}}$ for $i = 0, -1, -2$. We denote $\theta_{k,(i)}$ to be the i^{th} entry of $\theta_k \in \mathbb{R}^2$ and similarly for $u_k \in \mathbb{R}^2$.

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