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Deriving Euler's Equation for Rigid-Body Rotation via Lagrangian Dynamics with Generalized Coordinates

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Abstract: Euler's equation relates the change in angular momentum of a rigid body to the applied torque. This paper uses Lagrangian dynamics to derive Euler's equation in terms of generalized coordinates. This is done by parameterizing the angular velocity vector in terms of 3-2-1 and 3-1-3 Euler angles as well as Euler parameters, that is, quaternions. This paper fills a gap in the literature by using generalized coordinates to parameterize the angular velocity vector and thereby transform the dynamics obtained from Lagrangian dynamics into Euler's equation for rigid-body rotation.

Keywords: angular velocity; rotation; quaternions

MSC: 70A05; 70H03

1. Introduction

The rotational dynamics of a rigid body are modeled by Euler's equation [1] (p. 59), which relates the rate of change of the body angular momentum to the net torque. Let $\omega \in \mathbb{R}^3$ denote the angular velocity of the body relative to an inertial frame, let $J \in \mathbb{R}^{3\times 3}$ denote the inertia matrix of the body relative to its center of mass, and let τ denote the net torque applied to the body. All of these quantities are expressed in the body frame. Applying Newton–Euler dynamics yields Euler's equation

$$J\dot{\omega} + \omega \times J\omega = \tau. \tag{1}$$

An alternative approach to obtaining the dynamics of a mechanical system is to apply Hamilton's principle in the form of Lagrangian dynamics given by

$$\mathrm{d}_t \partial_{\dot{q}} T - \partial_q T = Q, \tag{2}$$

where T is the kinetic energy of the system, q is the vector of generalized coordinates, and Q is the vector of generalized forces arising from all external and dissipative forces and torques, including those arising from potential energy. Here, d_t denotes the total time derivative, and $\partial_{\dot{q}}$ and ∂_q denote the partial derivatives with respect to $\dot{q}$ and q, respectively.

For a mechanical system consisting of multiple rigid bodies, (2) obviates the need to determine conservative contact forces, which, in the absence of dissipative contact forces, circumvents the need for free-body analysis [2]. For the case of a single rigid body, however, (2) offers no advantage relative to a Newtonian-based derivation of Euler's equation. A Lagrangian-based derivation of Euler's equation is given in [3] (p. 281) using Lagrangian dynamics on Lie groups. As an alternative derivation of (1), the present note uses generalized coordinates within the context of classical Lagrangian dynamics. Related work includes [4,5], both of which use generalized coordinates to model the dynamics of

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linkages. The present paper extends [4,5] by deriving Euler's equation using both Euler angles and quaternions to parameterize the angular velocity vector. In particular, the present paper fills a gap in the literature by using generalized coordinates to parameterize the angular velocity vector and thereby transform the dynamics obtained from Lagrangian dynamics into Euler's equation for rigid-body rotation. Among all possible sequences consisting of three Euler-angle rotations, there are six that have three distinct axes and six that have the same first and last axes, for a total of twelve distinct sequences [6] (p. 764). Relabeling axes allows us to consider two representative sequences, namely, 3-2-1 (azimuth-elevation-bank) and 3-1-3 (precession-nutation-spin). These choices are commonly used for aircraft and spacecraft, respectively. As a further example, Euler parameters (quaternions) are also considered.

Notation: I_3 denotes the 3×3 identity matrix, and $A^{\mathsf{T}} \in \mathbb{R}^{l \times k}$ denotes the transpose of $A \in \mathbb{R}^{k \times l}$. For $x, y \in \mathbb{R}^3$, $x \times y$ denotes the cross product of x and y, and $x^{\times}$ denotes the cross-product matrix

$$x^{\times} \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \tag{3}$$

where $x = [x_1\ x_2\ x_3]^{\mathsf{T}}$, so that $x^{\times} y = x \times y$.

2. Preliminary Results

For a single rigid body, let $q = [q_1\ q_2\ q_3]^{\mathsf{T}} \in \mathbb{R}^3$ denote generalized coordinates, and assume that the angular velocity $\omega \in \mathbb{R}^3$ can be parameterized as

$$\omega(q, \dot{q}) = S(q)\dot{q}, \tag{4}$$

where $S(q) \in \mathbb{R}^{3 \times 3}$.

Assuming that the net force is zero and thus the center of mass of the body has zero inertial acceleration, it follows that

$$\begin{aligned} T(q, \dot{q}) &= \tfrac{1}{2}\omega(q, \dot{q})^{\mathsf{T}} J \omega(q, \dot{q}) \\ &= \tfrac{1}{2}\dot{q}^{\mathsf{T}} S(q)^{\mathsf{T}} J S(q) \dot{q}, \end{aligned} \tag{5}$$

and thus,

$$\partial_{\dot{q}} T(q, \dot{q}) = S(q)^{\mathsf{T}} J S(q) \dot{q}, \tag{6}$$

$$\mathrm{d}_t \partial_{\dot{q}} T(q, \dot{q}) = S(q)^{\mathsf{T}} J S(q) \ddot{q} + S(q)^{\mathsf{T}} J \dot{S}(q) \dot{q} + \dot{S}(q)^{\mathsf{T}} J S(q) \dot{q}, \tag{7}$$

$$\partial_q T(q, \dot{q}) = \begin{bmatrix} \dot{q}^{\mathsf{T}} [\partial_{q_1} S(q)]^{\mathsf{T}} J S(q) \dot{q} \\ \dot{q}^{\mathsf{T}} [\partial_{q_2} S(q)]^{\mathsf{T}} J S(q) \dot{q} \\ \dot{q}^{\mathsf{T}} [\partial_{q_3} S(q)]^{\mathsf{T}} J S(q) \dot{q} \end{bmatrix}. \tag{8}$$

Furthermore, it follows from [2] (8.10.6) that

$$Q = S(q)^{\mathsf{T}} \tau. \tag{9}$$

Now, combining (7)–(9) with (2) yields

$$S(q)^{\mathsf{T}} J S(q) \ddot{q} + S(q)^{\mathsf{T}} J \dot{S}(q) \dot{q} + \dot{S}(q)^{\mathsf{T}} J S(q) \dot{q} - \begin{bmatrix} \dot{q}^{\mathsf{T}} [\partial_{q_1} S(q)]^{\mathsf{T}} J S(q) \dot{q} \\ \dot{q}^{\mathsf{T}} [\partial_{q_2} S(q)]^{\mathsf{T}} J S(q) \dot{q} \\ \dot{q}^{\mathsf{T}} [\partial_{q_3} S(q)]^{\mathsf{T}} J S(q) \dot{q} \end{bmatrix} = S(q)^{\mathsf{T}} \tau. \tag{10}$$

If $S(q)$ is non-singular, then

$$JS(q)\ddot{q} + J\dot{S}(q)\dot{q} + S(q)^{-\mathsf{T}}\dot{S}(q)^{\mathsf{T}}JS(q)\dot{q} - S(q)^{-\mathsf{T}}\begin{bmatrix}\dot{q}^{\mathsf{T}}[\partial_{q_1}S(q)]^{\mathsf{T}}JS(q)]\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_2}S(q)]^{\mathsf{T}}JS(q)\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_3}S(q)]^{\mathsf{T}}JS(q)\dot{q}\end{bmatrix} = \tau, \tag{11}$$

which can be viewed as Euler's equation expressed in terms of arbitrary generalized coordinates.

Next, noting that

$$\dot{\omega}(q,\dot{q}) = S(q)\ddot{q} + \dot{S}(q)\dot{q}, \tag{12}$$

Equation (11) can be written as

$$J\dot{\omega}(q,\dot{q}) + S(q)^{-\mathsf{T}}\left(\dot{S}(q)^{\mathsf{T}}JS(q)\dot{q} - \begin{bmatrix}\dot{q}^{\mathsf{T}}[\partial_{q_1}S(q))]^{\mathsf{T}}JS(q)]\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_2}S(q)]^{\mathsf{T}}JS(q)\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_3}S(q)]^{\mathsf{T}}JS(q)\dot{q}\end{bmatrix}\right) = \tau. \tag{13}$$

Comparing (13) with Euler's Equation (1) written in terms of the angular velocity implies

$$S(q)^{-\mathsf{T}}\left(\dot{S}(q)^{\mathsf{T}}JS(q)\dot{q} - \begin{bmatrix}\dot{q}^{\mathsf{T}}[\partial_{q_1}S(q)]^{\mathsf{T}}JS(q)\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_2}S(q)]^{\mathsf{T}}JS(q)\dot{q}\\ \dot{q}^{\mathsf{T}}[\partial_{q_3}S(q)]^{\mathsf{T}}JS(q)\dot{q}\end{bmatrix}\right) = \omega(q,\dot{q}) \times J\omega(q,\dot{q}). \tag{14}$$

Our objective is to verify this identity for rotations parameterized by Euler angles and Euler parameters (quaternions).

For the following result, the columns of $S(q)$ are denoted by $S_1(q)$, $S_2(q)$, and $S_3(q)$ so that

$$S(q) = [S_1(q)\ \ S_2(q)\ \ S_3(q)]. \tag{15}$$

We note that **(a)** is given by Equation (A24) of [7].

Proposition 1. *Define S by (4). Then, the following properties are equivalent:*

(a) *For all q and $\dot{q}$,*

$$\dot{S}(q) + [S(q)\dot{q}]^{\times}S(q) = [\partial_{q_1}S(q)\dot{q}\ \partial_{q_2}S(q)\dot{q}\ \partial_{q_3}S(q)\dot{q}]. \tag{16}$$

(b) *For all q and $\dot{q}$,*

$$\begin{aligned}\sum_{i=1}^{3}\dot{q}_i\partial_{q_i}S(q) + [S_2(q)\times S_3(q)\quad S_3(q)\times S_1(q)\quad S_1(q)\times S_2(q)]\dot{q}^{\times}\\ = [\partial_{q_1}S(q)\dot{q}\quad \partial_{q_2}S(q)\dot{q}\quad \partial_{q_3}S(q)\dot{q}].\end{aligned} \tag{17}$$

(c) *For all q,*

$$\partial_{q_2}S_1(q) - \partial_{q_1}S_2(q) = S_1(q)\times S_2(q), \tag{18}$$

$$\partial_{q_3}S_1(q) - \partial_{q_1}S_3(q) = S_1(q)\times S_3(q), \tag{19}$$

$$\partial_{q_3}S_2(q) - \partial_{q_2}S_3(q) = S_2(q)\times S_3(q). \tag{20}$$

Now, assume that $S(q)$ is non-singular. Then, (a)–(c) are equivalent to

$$S(q)^{\mathsf{T}}[\partial_{q_3}S_2(q) - \partial_{q_2}S_3(q)\quad \partial_{q_1}S_3(q) - \partial_{q_3}S_1(q)\quad \partial_{q_2}S_1(q) - \partial_{q_1}S_2(q)] = \det(S(q))I_3. \tag{21}$$

The following lemmas are needed.

Lemma 1. *Let $x \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3\times 3}$. Then,*

$$A^{\mathsf{T}}(Ax)^{\times}A = (\det A)x^{\times}. \tag{22}$$

Proof. of Fact 4.12.1 in [8] (p. 385). □

Lemma 2. *Let $A = [A_1 \;\; A_2 \;\; A_3] \in \mathbb{R}^{3\times 3}$. Then,*

$$A^{\mathsf{T}}[A_2 \times A_3 \quad A_3 \times A_1 \quad A_1 \times A_2] = (\det A)I_3. \tag{23}$$

Now, let $x \in \mathbb{R}^3$. Then,

$$[A_2 \times A_3 \quad A_3 \times A_1 \quad A_1 \times A_2]x^{\times} = (Ax)^{\times}A. \tag{24}$$

Proof. of Fact 4.12.1 in [8] (p. 385). In the case where A is non-singular, the second statement follows from (22) and (23). In the case where A is singular, the conclusion follows by continuity since both sides of (24) are continuous functions of the columns (A_1, A_2, A_3) of A and the set of non-singular matrices is dense in $\mathbb{R}^{3\times 3}$. □

Proof of Proposition 1. Note that

$$\dot{S}(q) = \sum_{i=1}^{3} \dot{q}_i \partial_{q_i} S(q). \tag{25}$$

Furthermore, it follows from (24) that:

$$[S(q)\dot{q}]^{\times}S(q) = [S_2(q) \times S_3(q) \quad S_3(q) \times S_1(q) \quad S_1(q) \times S_2(q)]\dot{q}^{\times}. \tag{26}$$

Therefore, (25) and (26) imply that **(a)** and **(b)** are equivalent.

To prove that **(b)** and **(c)** are equivalent, note that **(b)** is equivalent to $L(\dot{q}) = R(\dot{q})$ for all $\dot{q} \in \mathbb{R}^3$, where L and R are the linear operators defined for all $x = [x_1 \;\; x_2 \;\; x_3]^{\mathsf{T}} \in \mathbb{R}^3$ by

$$L(x) = [\partial_{q_1} S(q)x \quad \partial_{q_2} S(q)x \quad \partial_{q_3} S(q)x] - \sum_{i=1}^{3} x_i \partial_{q_i} S(q), \tag{27}$$

$$R(x) = [S_2(q) \times S_3(q) \quad S_3(q) \times S_1(q) \quad S_1(q) \times S_2(q)]x^{\times}. \tag{28}$$

Since R and L are linear, it follows that $L(\dot{q}) = R(\dot{q})$ for all $\dot{q} \in \mathbb{R}^3$ if and only if:

$$L(e_i) = R(e_i), \qquad \text{for all } i = 1, 2, 3, \tag{29}$$

where $e_1 = [1 \;\; 0 \;\; 0]^{\mathsf{T}}$, $e_2 = [0 \;\; 1 \;\; 0]^{\mathsf{T}}$, and $e_3 = [0 \;\; 0 \;\; 1]^{\mathsf{T}}$ because (e_1, e_2, e_3) is a basis of $\mathbb{R}^3$.

Next, note that

$$\begin{aligned} L(e_1) &= [\partial_{q_1} S_1(q) \;\; \partial_{q_2} S_1(q) \;\; \partial_{q_3} S_1(q)] - [\partial_{q_1} S_1(q) \;\; \partial_{q_1} S_2(q) \;\; \partial_{q_1} S_3(q)] \\ &= [0 \quad \partial_{q_2} S_1(q) - \partial_{q_1} S_2(q) \quad \partial_{q_3} S_1(q) - \partial_{q_1} S_3(q)], \end{aligned} \tag{30}$$

$$L(e_2) = [\partial_{q_1} S_2(q) - \partial_{q_2} S_1(q) \quad 0 \quad \partial_{q_3} S_2(q) - \partial_{q_2} S_3(q)], \tag{31}$$

$$L(e_3) = [\partial_{q_1} S_3(q) - \partial_{q_3} S_1(q) \quad \partial_{q_2} S_3(q) - \partial_{q_3} S_2(q) \quad 0], \tag{32}$$

and

$$R(e_1) = [S_2(q) \times S_3(q) \quad S_3(q) \times S_1(q) \quad S_1(q) \times S_2(q)][0 \quad e_3 \quad -e_2],$$
$$= [0 \;\; S_1(q) \times S_2(q) \;\; S_1(q) \times S_3(q)], \tag{33}$$

$$R(e_2) = [S_2(q) \times S_1(q) \;\; 0 \;\; S_2(q) \times S_3(q)], \tag{34}$$

$$R(e_3) = [S_3(q) \times S_1(q) \quad S_3(q) \times S_2(q) \quad 0]. \tag{35}$$

Comparing (30)–(32) with (33)–(34) shows that (29) is equivalent to **(c)**. Finally, (21) follows from (18)–(20) and (22). □

To demonstrate the relevance of (14)–(16), note that transposing and rearranging (16) yields:

$$\dot{S}(q)^\mathsf{T} - \begin{bmatrix} \dot{q}^\mathsf{T}[\partial_{q_1} S(q)]^\mathsf{T} \\ \dot{q}^\mathsf{T}[\partial_{q_2} S(q)]^\mathsf{T} \\ \dot{q}^\mathsf{T}[\partial_{q_3} S(q)]^\mathsf{T} \end{bmatrix} = S(q)^\mathsf{T}[S(q)\dot{q}]^\times, \tag{36}$$

and thus, assuming that $S(q)$ is non-singular,

$$S(q)^{-\mathsf{T}} \left(\dot{S}(q)^\mathsf{T} - \begin{bmatrix} \dot{q}^\mathsf{T}[\partial_{q_1} S(q)]^\mathsf{T} \\ \dot{q}^\mathsf{T}[\partial_{q_2} S(q)]^\mathsf{T} \\ \dot{q}^\mathsf{T}[\partial_{q_3} S(q)]^\mathsf{T} \end{bmatrix} \right) = [S(q)\dot{q}]^\times. \tag{37}$$

Finally, multiplying (37) on the right by $JS(q)\dot{q}$ yields

$$S(q)^{-\mathsf{T}} \left(\dot{S}(q)^\mathsf{T} JS(q)\dot{q} - \begin{bmatrix} \dot{q}^\mathsf{T}[\partial_{q_1} S(q)]^\mathsf{T} JS(q)\dot{q} \\ \dot{q}^\mathsf{T}[\partial_{q_2} S(q)]^\mathsf{T} JS(q)\dot{q} \\ \dot{q}^\mathsf{T}[\partial_{q_3} S(q)]^\mathsf{T} JS(q)\dot{q} \end{bmatrix} \right) = \omega(q,\dot{q}) \times J\omega(q,\dot{q}), \tag{38}$$

which is precisely (14). □

For a given choice of q, it is easier to verify (18)–(20) than (16) or (17). In the next three sections, (18)–(20) are verified for 3-2-1 and 3-1-3 Euler angles as well as Euler parameters (quaternions).

3. Verification of (18)–(20) for 3-2-1 Euler Angles

Letting (Ψ, Θ, Φ) denote 3-2-1 (azimuth-elevation-bank) Euler angles, it follows that

$$\omega(q,\dot{q}) = S(\Phi,\Theta)\dot{q}, \tag{39}$$

where

$$S(\Phi,\Theta) = [S_1 \quad S_2(\Phi) \quad S_3(\Phi,\Theta)] \tag{40}$$

$$= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}, \tag{41}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}. \tag{42}$$

Note that $\det S(\Phi,\Theta) = \cos\Theta$, and thus $S(\Phi,\Theta)$ is singular if and only if gimbal lock occurs. Hence,

$$\begin{aligned}
\partial_{q_2} S_1(q) - \partial_{q_1} S_2(q) &= -\partial_\Phi S_2(\Phi) = \begin{bmatrix} 0 \\ \sin\Phi \\ \cos\Phi \end{bmatrix} = S_1 \times S_2(\Phi), \\
\partial_{q_3} S_1(q) - \partial_{q_1} S_3(q) &= -\partial_\Phi S_3(\Phi,\Theta) = \begin{bmatrix} 0 \\ -\cos\Phi\,\cos\Theta \\ \sin\Phi\,\cos\Theta \end{bmatrix} = S_1 \times S_3(\Phi,\Theta), \qquad (43) \\
\partial_{q_3} S_2(q) - \partial_{q_2} S_3(q) &= -\partial_\Theta S_3(\Phi,\Theta) = \begin{bmatrix} \cos\Theta \\ \sin\Phi\,\sin\Theta \\ \cos\Phi\,\sin\Theta \end{bmatrix} = S_2(\Phi) \times S_3(\Phi,\Theta).
\end{aligned}$$

Hence, (18)–(20) hold, and thus (16) and (17) are verified.

4. Verification of (18)–(20) for 3-1-3 Euler Angles

Letting (Φ,Θ,Ψ) denote 3-1-3 (precession-nutation-spin) Euler angles, it follows that

$$\omega(q,\dot{q}) = S(\Psi,\Theta)\dot{q}, \qquad (44)$$

where

$$S(\Psi,\Theta) = [S_1 \quad S_2(\Psi) \quad S_3(\Psi,\Theta)] \qquad (45)$$

$$= \begin{bmatrix} 0 & \cos\Psi & (\sin\Psi)\sin\Theta \\ 0 & -\sin\Psi & (\cos\Psi)\sin\Theta \\ 1 & 0 & \cos\Theta \end{bmatrix}, \qquad (46)$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \begin{bmatrix} \Psi \\ \Theta \\ \Phi \end{bmatrix}. \qquad (47)$$

Note that $\det S(\Psi,\Theta) = \sin\Theta$, and thus $S(\Psi,\Theta)$ is singular if and only if gimbal lock occurs. Hence,

$$\begin{aligned}
\partial_{q_2} S_1(q) - \partial_{q_1} S_2(q) &= -\partial_\Psi S_2(\Psi) = \begin{bmatrix} \sin\Psi \\ \cos\Psi \\ 0 \end{bmatrix} = S_1 \times S_2(\Psi), \\
\partial_{q_3} S_1(q) - \partial_{q_1} S_3(q) &= -\partial_\Psi S_3(\Psi,\Theta) = \begin{bmatrix} -\cos\Psi\,\sin\Theta, \\ \sin\Psi\,\sin\Theta \\ 0 \end{bmatrix} = S_1 \times S_3(\Psi,\Theta), \qquad (48) \\
\partial_{q_3} S_2(q) - \partial_{q_2} S_3(q) &= -\partial_\Theta S_3(\Psi,\Theta) = \begin{bmatrix} -\sin\Psi\,\cos\Theta \\ -\cos\Psi\,\cos\Theta \\ \sin\Theta \end{bmatrix} = S_2(\Psi) \times S_3(\Psi,\Theta).
\end{aligned}$$

Hence, (18)–(20) hold, and thus (16) and (17) are verified.

5. Verification of (21) for Euler Parameters

To avoid gimbal lock, an alternative approach is to use Euler parameters (quaternions). In this case,

$$\tilde{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\theta \\ (\sin\frac{1}{2}\theta)n \end{bmatrix}, \qquad (49)$$

where $\theta \in (-\pi, \pi]$ is the eigenangle and $n \in \mathbb{R}^3$ is the unit eigenaxis. Since $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, it follows that $q_1 = \sqrt{1 - q_2^2 - q_3^2 - q_4^2}$, and thus the generalized coordinates are $q = [q_2 \ \ q_3 \ \ q_4]^\mathsf{T}$. With this notation, assuming that $\theta \neq \pi$ and thus $q_1 > 0$, it follows that (4) holds with

$$S(q) = 2\begin{bmatrix} q_1 + \dfrac{q_2^2}{q_1} & q_4 + \dfrac{q_2 q_3}{q_1} & -q_3 + \dfrac{q_2 q_4}{q_1} \\ -q_4 + \dfrac{q_2 q_3}{q_1} & q_1 + \dfrac{q_3^2}{q_1} & q_2 + \dfrac{q_3 q_4}{q_1} \\ q_3 + \dfrac{q_2 q_4}{q_1} & -q_2 + \dfrac{q_3 q_4}{q_1} & q_1 + \dfrac{q_4^2}{q_1} \end{bmatrix}. \tag{50}$$

Next, note that, for all $i = 2, 3, 4$, $\partial_{q_i} q_1 = -q_i / q_1$. Thus,

$$\partial_{q_3} S_1(q) - \partial_{q_2} S_2(q) = 2\begin{bmatrix} -\dfrac{q_3}{q_1} + \dfrac{q_2^2 q_3}{q_1^3} - \dfrac{q_3}{q_1} - \dfrac{q_2^2 q_3}{q_1^3} \\ \dfrac{q_2}{q_1} + \dfrac{q_2 q_3^2}{q_1^3} + \dfrac{q_2}{q_1} - \dfrac{q_3^2 q_2}{q_1^3} \\ 1 + \dfrac{q_2 q_3 q_4}{q_1^3} + 1 - \dfrac{q_2 q_3 q_4}{q_1^3} \end{bmatrix} = \frac{4}{q_1}\begin{bmatrix} -q_3 \\ q_2 \\ q_1 \end{bmatrix}, \tag{51}$$

$$\partial_{q_4} S_1(q) - \partial_{q_2} S_3(q) = 2\begin{bmatrix} -\dfrac{q_4}{q_1} + \dfrac{q_2^2 q_4}{q_1^3} - \dfrac{q_4}{q_1} - \dfrac{q_2^2 q_4}{q_1^3} \\ -1 + \dfrac{q_2 q_3 q_4}{q_1^3} - 1 - \dfrac{q_2 q_3 q_4}{q_1^3} \\ \dfrac{q_2}{q_1} + \dfrac{q_2 q_4^2}{q_1^3} + \dfrac{q_2}{q_1} - \dfrac{q_2 q_4^2}{q_1^3} \end{bmatrix} = \frac{4}{q_1}\begin{bmatrix} -q_4 \\ -q_1 \\ q_2 \end{bmatrix}, \tag{52}$$

$$\partial_{q_4} S_2(q) - \partial_{q_3} S_3(q) = 2\begin{bmatrix} 1 + \dfrac{q_2 q_3 q_4}{q_1^3} + 1 - \dfrac{q_2 q_3 q_4}{q_1^3} \\ -\dfrac{q_4}{q_1} + \dfrac{q_3^2 q_4}{q_1^3} - \dfrac{q_4}{q_1} - \dfrac{q_3^2 q_4}{q_1^3} \\ \dfrac{q_3}{q_1} + \dfrac{q_3 q_4^2}{q_1^3} + \dfrac{q_3}{q_1} - \dfrac{q_3 q_4^2}{q_1^3} \end{bmatrix} = \frac{4}{q_1}\begin{bmatrix} q_1 \\ -q_4 \\ q_3 \end{bmatrix}. \tag{53}$$

Thus,

$$\begin{aligned} M(q) &\triangleq [\partial_{q_4} S_2(q) - \partial_{q_3} S_3(q) \ \ \partial_{q_2} S_3(q) - \partial_{q_4} S_1(q) \ \ \partial_{q_3} S_1(q) - \partial_{q_2} S_2(q)] \\ &= \frac{4}{q_1}\begin{bmatrix} q_1 & q_4 & -q_3 \\ -q_4 & q_1 & q_2 \\ q_3 & -q_2 & q_1 \end{bmatrix}, \end{aligned} \tag{54}$$

and using $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$ yields

$$M(q)^{\mathsf{T}} S(q) = \frac{8}{q_1} \begin{bmatrix} q_1 & -q_4 & q_3 \\ q_4 & q_1 & -q_2 \\ -q_3 & q_2 & q_1 \end{bmatrix} \begin{bmatrix} q_1 + \frac{q_2^2}{q_1} & q_4 + \frac{q_2 q_3}{q_1} & -q_3 + \frac{q_2 q_4}{q_1} \\ -q_4 + \frac{q_2 q_3}{q_1} & q_1 + \frac{q_3^2}{q_1} & q_2 + \frac{q_3 q_4}{q_1} \\ q_3 + \frac{q_2 q_4}{q_1} & -q_2 + \frac{q_3 q_4}{q_1} & q_1 + \frac{q_4^2}{q_1} \end{bmatrix}$$

$$= \frac{8}{q_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{55}$$

Since $\det(M(q)) = 64/q_1^2$, (55) implies that $\det(S(q)) = 8/q_1$. Thus, $S(q)$ is non-singular and satisfies (21). Consequently, (16) and (17) are verified for Euler parameters (quaternions).

6. Conclusions

For a single unconstrained rigid body, this paper filled a gap in the literature by using generalized coordinates to parameterize the angular velocity vector and thereby transform the dynamics obtained from Lagrangian dynamics into Euler's equation for rigid-body rotation. The derivation, which relies on matrix techniques and cross-product identities, strengthens the connection between Lagrangian and Newton–Euler dynamics.

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