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Retrospective-Cost-Based Model Reference Adaptive Control of Nonminimum-Phase Systems

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Abstract— **This paper presents a novel approach to model reference adaptive control inspired by the adaptive pole-placement technique of Elliot and based on retrospective cost optimization (RC-MRAC). RC-MRAC is applicable to nonminimum-phase (NMP) systems assuming that the NMP zeros are known. Under this assumption, the advantage of RC-MRAC is a reduced need for persistency.**

I. Introduction

The objective of model reference adaptive control (MRAC) is to have the output of an uncertain system follow the response of a given reference system. The literature on MRAC and its applications is vast and varied, for example, [1], [2], [3], [4]. MRAC methods can be divided into two categories, namely, indirect and direct. Indirect MRAC uses system identification followed by controller adaptation using the identified model, whereas direct MRAC adapts the controller using limited modeling information. Both types of methods typically use either gradient descent or recursive least squares for the adaptation [5], [6], [7], [8]. MRAC methods have been extensively developed, including extensions to nonminimum-phase (NMP) and nonlinear systems [9], [10], [11], [12], [13].

The present paper develops a novel MRAC technique based on retrospective cost adaptive control (RCAC). RCAC is a direct adaptive control method for command following and disturbance rejection for systems with uncertain dynamics and disturbance spectra [14]. For SISO discrete-time or sampled-data systems, RCAC requires knowledge of the sign of the leading numerator coefficient, relative degree, and NMP zeros. RCAC minimizes a retrospective performance measure based on the difference between filtered past control inputs and filtered, re-optimized past control inputs. In order to further reduce the dependence on prior modeling, an indirect adaptive control extension of RCAC was developed in [15].

Retrospective cost model reference adaptive control (RC-MRAC) was developed in [16] with stability analysis given in [17]. A related technique was developed in [18]. As in the case of RCAC, RC-MRAC is applicable to discrete-time and sampled-data systems with known NMP zeros; minimum-phase zeros need not be known.

The version of RC-MRAC developed in the present paper is inspired by the adaptive pole-placement algorithm developed by Elliott [9], [10]. The remarkable feature of the approach of [9], [10] is its applicability to NMP systems with *unknown* NMP zeros. The drawback of this technique is the need for sufficient persistency in order to achieve command following, even for step commands. Although this requirement was alleviated in [19] through the use of DREM, the need for persistency is nontrivial.

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The goal and contribution of the present paper is to develop RC-MRAC and assess its performance from the perspective of both command following and adaptive pole-placement. Numerical examples show that, in contrast to [19], RC-MRAC does not require persistency. The price paid for alleviating the need for persistency is knowledge of the NMP zeros.

The structure of the paper is as follows. Section II gives an overview of the MRAC problem. Section III gives the development of RC-MRAC. Section IV provides examples of RC-MRAC for a variety of systems, including minimum-phase and NMP, and an example extending the algorithm to disturbance rejection. Additionally, a comparison to Elliot's adaptive pole-placement controller for a case where persistency is minimal is given.

II. Model Reference Adaptive Control

Consider the discrete-time SISO system

$$y_k = \frac{N(\mathbf{q}^{-1})}{D(\mathbf{q}^{-1})} u_k, \tag{1}$$

where

$$N(\mathbf{q}^{-1}) \triangleq \sum_{i=n_{\mathrm{r}}}^{n} N_i \mathbf{q}^{-i}, \tag{2}$$

$$D(\mathbf{q}^{-1}) \triangleq 1 + \sum_{i=1}^{n} D_i \mathbf{q}^{-i}, \tag{3}$$

are coprime, $N_{n_{\mathrm{r}}} \neq 0$, and n_{r} is the relative degree of $\frac{N(\mathbf{q}^{-1})}{D(\mathbf{q}^{-1})}$ as a rational function of $\mathbf{q}$. In the model reference adaptive control (MRAC) problem, the goal is to find a controller $G_{\mathrm{c}}(\mathbf{q}^{-1})$ such that the output y_k follows the desired reference response $y_{\mathrm{m},k}$ to a command r_k given by

$$y_{\mathrm{m},k} = \frac{N_{\mathrm{m}}(\mathbf{q}^{-1})}{D_{\mathrm{m}}(\mathbf{q}^{-1})} r_k, \tag{4}$$

where

$$N_{\mathrm{m}}(\mathbf{q}^{-1}) \triangleq \sum_{i=n_{\mathrm{r}}}^{n} N_{\mathrm{m},i} \mathbf{q}^{-i}, \tag{5}$$

$$D_{\mathrm{m}}(\mathbf{q}^{-1}) \triangleq 1 + \sum_{i=1}^{n} D_{\mathrm{m},i} \mathbf{q}^{-i}. \tag{6}$$

As shown in Figure 1, the error e_k between the actual system response y_k and the reference model response $y_{\mathrm{m},k}$ is used to update the controller. The direct MRAC problem differs from the indirect case in that the system is not identified, but knowledge of the NMP zeros of (1) is typically needed to prevent unstable pole-zero cancellation.

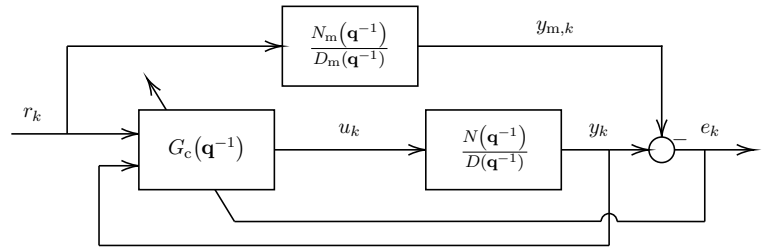

Fig. 1. Block diagram of the direct model reference adaptive control problem.

III. Retrospective Cost Model Reference Adaptive Control (RC-MRAC)

A. RC-MRAC Development

Defining

$$x_k \triangleq \frac{1}{D(\mathbf{q}^{-1})} u_k, \tag{7}$$

which satisfies

$$D(\mathbf{q}^{-1}) x_k = u_k, \tag{8}$$

it follows that (1) can be written as

$$y_k = N(\mathbf{q}^{-1}) x_k. \tag{9}$$

Let $N(\mathbf{q}^{-1})$ be factored as

$$N(\mathbf{q}^{-1}) = N_{n_{\rm r}} N_{\rm u}(\mathbf{q}^{-1}) N_{\rm s}(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}}, \tag{10}$$

where $N_{\rm u}(\mathbf{q}^{-1})$ and $N_{\rm s}(\mathbf{q}^{-1})$ as a function of $\mathbf{q}$ are monic polynomials of order $n_{\rm u}$ and $n_{\rm s}$ whose roots have modulus at least 1 and less than 1, respectively. Next, consider the controller

$$u_k = N_{\rm c}(\mathbf{q}^{-1}) y_k + D_{\rm c}(\mathbf{q}^{-1}) u_k + R_{\rm c}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1}) r_k, \tag{11}$$

where

$$N_{\rm c}(\mathbf{q}^{-1}) \triangleq \sum_{i=1}^{n} N_{{\rm c},i} \mathbf{q}^{-i}, \tag{12}$$

$$D_{\rm c}(\mathbf{q}^{-1}) \triangleq \sum_{i=1}^{n} D_{{\rm c},i} \mathbf{q}^{-i}, \tag{13}$$

$$R_{\rm c}(\mathbf{q}^{-1}) \triangleq R_{{\rm c},0} + \sum_{i=1}^{n_{\rm s}} R_{{\rm c},i} \mathbf{q}^{-i}, \tag{14}$$

$$F(\mathbf{q}^{-1}) \triangleq 1 + \sum_{i=1}^{n-n_{\rm s}} F_i \mathbf{q}^{-i}, \tag{15}$$

and $F(\mathbf{q}^{-1})$ is an arbitrary stable monic polynomial in $\mathbf{q}$ of order $n-n_{\rm s}$. Combining (8), (9), and (11) yields

$$\begin{aligned} D(\mathbf{q}^{-1}) x_k = {} & N_{\rm c}(\mathbf{q}^{-1}) N(\mathbf{q}^{-1}) x_k + D_{\rm c}(\mathbf{q}^{-1}) D(\mathbf{q}^{-1}) x_k \\ & + R_{\rm c}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1}) r_k, \end{aligned} \tag{16}$$

which implies

$$x_k = \frac{R_{\rm c}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1})}{\tilde{D}(\mathbf{q}^{-1})} r_k, \tag{17}$$

where

$$\tilde{D}(\mathbf{q}^{-1}) \triangleq D(\mathbf{q}^{-1}) - N_{\rm c}(\mathbf{q}^{-1}) N(\mathbf{q}^{-1}) - D_{\rm c}(\mathbf{q}^{-1}) D(\mathbf{q}^{-1}). \tag{18}$$

Proposition 3.1: Let the desired closed-loop poles be the roots of

$$D_{\rm m}(\mathbf{q}^{-1}) = 1 + \sum_{i=1}^{n} D_{{\rm m},i} \mathbf{q}^{-i}, \tag{19}$$

and assume there exist $N_{\rm c}^*(\mathbf{q}^{-1})$ and $D_{\rm c}^*(\mathbf{q}^{-1})$ such that

$$D_{\rm m}(\mathbf{q}^{-1}) N_{\rm s}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1}) = \tilde{D}^*(\mathbf{q}^{-1}), \tag{20}$$

where

$$\begin{aligned} \tilde{D}^*(\mathbf{q}^{-1}) \triangleq {} & D(\mathbf{q}^{-1}) - N_{\rm c}^*(\mathbf{q}^{-1}) N(\mathbf{q}^{-1}) \\ & - D_{\rm c}^*(\mathbf{q}^{-1}) D(\mathbf{q}^{-1}). \end{aligned} \tag{21}$$

Then, the closed-loop dynamics are given by

$$y_k = \frac{N_{n_{\rm r}} N_u(\mathbf{q}^{-1}) R_{\rm c}(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}}}{D_{\rm m}(\mathbf{q}^{-1})} r_k. \tag{22}$$

Proof: Using (9), (17) with $\tilde{D}(\mathbf{q}^{-1}) = \tilde{D}^*(\mathbf{q}^{-1})$ and (20) yields

$$\begin{aligned} y_k = N(\mathbf{q}^{-1}) x_k &= \frac{N(\mathbf{q}^{-1}) R_{\rm c}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1})}{\tilde{D}^*(\mathbf{q}^{-1})} r_k \\ &= \frac{N(\mathbf{q}^{-1}) R_{\rm c}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1})}{D_{\rm m}(\mathbf{q}^{-1}) N_{\rm s}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1})} r_k \\ &= \frac{N_{n_{\rm r}} N_u(\mathbf{q}^{-1}) R_{\rm c}(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}}}{D_{\rm m}(\mathbf{q}^{-1})} r_k. \end{aligned} \qquad \square$$

For later use, note that multiplying both sides of (21) by x_k, and using (8), (9), and (20) yields

$$\begin{aligned} & D_{\rm m}(\mathbf{q}^{-1}) N_{\rm s}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1}) x_k \\ & \quad = u_k - N_{\rm c}^*(\mathbf{q}^{-1}) y_k - D_{\rm c}^*(\mathbf{q}^{-1}) u_k. \end{aligned} \tag{23}$$

Proposition 3.2: Assume there exists $R_{\rm c}^*(\mathbf{q}^{-1})$ such that

$$N_{\rm m}(\mathbf{q}^{-1}) = N_{n_{\rm r}} N_u(\mathbf{q}^{-1}) R_{\rm c}^*(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}}, \tag{24}$$

and define

$$\tilde{N}_{\rm c}(\mathbf{q}^{-1}) \triangleq \hat{N}_{\rm c}(\mathbf{q}^{-1}) - N_{\rm c}^*(\mathbf{q}^{-1}), \tag{25}$$

$$\tilde{D}_{\rm c}(\mathbf{q}^{-1}) \triangleq \hat{D}_{\rm c}(\mathbf{q}^{-1}) - D_{\rm c}^*(\mathbf{q}^{-1}), \tag{26}$$

$$\tilde{R}_{\rm c}(\mathbf{q}^{-1}) \triangleq \hat{R}_{\rm c}(\mathbf{q}^{-1}) - R_{\rm c}^*(\mathbf{q}^{-1}). \tag{27}$$

Then,

$$\begin{aligned} & N_{n_{\rm r}} N_u(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}} [\tilde{N}_{\rm c}(\mathbf{q}^{-1}) y_k + \tilde{D}_{\rm c}(\mathbf{q}^{-1}) u_k + \tilde{R}_{\rm c}(\mathbf{q}^{-1}) r_k] \\ & \quad = D_{\rm m}(\mathbf{q}^{-1}) F(\mathbf{q}^{-1}) (y_k - y_{{\rm m},k}) \\ & \qquad - N_{n_{\rm r}} N_u(\mathbf{q}^{-1}) \mathbf{q}^{-n_{\rm r}} [u_k - \hat{N}_{\rm c}(\mathbf{q}^{-1}) y_k \\ & \qquad\qquad - \hat{D}_{\rm c}(\mathbf{q}^{-1}) u_k - \hat{R}_{\rm c}(\mathbf{q}^{-1}) r_k]. \end{aligned} \tag{28}$$

Proof: Multiplying both sides of (23) by $N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}$ and using (9) yields

$$\begin{aligned}&D_\mathrm{m}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})y_k\\&\quad=N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}\left[u_k-N_\mathrm{c}^*(\mathbf{q}^{-1})y_k-D_\mathrm{c}^*(\mathbf{q}^{-1})u_k\right].\end{aligned}\tag{29}$$

Subtracting $F(\mathbf{q}^{-1})N_\mathrm{m}(\mathbf{q}^{-1})r_k$ from both sides of (29) and using (4) yields

$$\begin{aligned}&D_\mathrm{m}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})(y_k-y_{\mathrm{m},k})\\&\quad=N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}\left[u_k-N_\mathrm{c}^*(\mathbf{q}^{-1})y_k-D_\mathrm{c}^*(\mathbf{q}^{-1})u_k\right]\\&\qquad-F(\mathbf{q}^{-1})N_\mathrm{m}(\mathbf{q}^{-1})r_k.\end{aligned}\tag{30}$$

Then combining (24) with (30) yields

$$\begin{aligned}&D_\mathrm{m}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})(y_k-y_{\mathrm{m},k})\\&\quad-N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}[u_k-N_\mathrm{c}^*(\mathbf{q}^{-1})y_k-D_\mathrm{c}^*(\mathbf{q}^{-1})u_k\\&\qquad-R_\mathrm{c}^*(\mathbf{q}^{-1})F(\mathbf{q}^{-1})r_k]=0.\end{aligned}\tag{31}$$

Finally, substituting (25)-(27) into (31) yields (28). □

B. RC-MRAC Algorithm

Note that all the terms on the right-hand side of (28) are known, and thus the sum of terms on the left-hand side is known despite the fact that $\tilde{N}_\mathrm{c}(\mathbf{q}^{-1})$, $\tilde{D}_\mathrm{c}(\mathbf{q}^{-1})$, and $\tilde{R}_\mathrm{c}(\mathbf{q}^{-1})$ are individually unknown. Furthermore, if (25)-(27) are all zero, then both sides of (28) are zero. We thus define the performance variable

$$\begin{aligned}z_k\triangleq&D_\mathrm{m}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})(y_k-y_{\mathrm{m},k})\\&-N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}[u_k-\hat{N}_\mathrm{c}(\mathbf{q}^{-1})y_k-\hat{D}_\mathrm{c}(\mathbf{q}^{-1})u_k\\&\qquad-\hat{R}_\mathrm{c}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})r_k]\end{aligned}\tag{32}$$

$$\begin{aligned}=&N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}[\tilde{N}_\mathrm{c}(\mathbf{q}^{-1})y_k+\tilde{D}_\mathrm{c}(\mathbf{q}^{-1})u_k\\&+\tilde{R}_\mathrm{c}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})r_k].\end{aligned}\tag{33}$$

Note that, if $\tilde{N}_\mathrm{c}(\mathbf{q}^{-1})$, $\tilde{D}_\mathrm{c}(\mathbf{q}^{-1})$, and $\tilde{R}_\mathrm{c}(\mathbf{q}^{-1})$ are all zero, then z_k is zero. We thus seek estimates $\hat{N}_\mathrm{c}(\mathbf{q}^{-1})$, $\hat{D}_\mathrm{c}(\mathbf{q}^{-1})$, and $\hat{R}_\mathrm{c}(\mathbf{q}^{-1})$ of $N_\mathrm{c}^*(\mathbf{q}^{-1})$, $D_\mathrm{c}^*(\mathbf{q}^{-1})$, and $R_\mathrm{c}^*(\mathbf{q}^{-1})$, respectively, that minimize the magnitude of z_k.

Proposition 3.3: Define

$$\begin{aligned}\theta\triangleq[&N_{\mathrm{c},1}^*\quad\cdots\quad N_{\mathrm{c},n}^*\quad D_{\mathrm{c},1}^*\quad\cdots\quad D_{\mathrm{c},n}^*\\&R_{\mathrm{c},0}^*\quad\cdots\quad R_{\mathrm{c},n_\mathrm{s}}^*]^\mathrm{T},\end{aligned}\tag{34}$$

then

$$z_{\mathrm{f},k}-u_{\mathrm{f},k}+\Phi_{\mathrm{f},k}\theta=0,\tag{35}$$

where

$$r_{\mathrm{f},k}\triangleq F(\mathbf{q}^{-1})r_k\tag{36}$$

$$\begin{aligned}\Phi_k\triangleq[&y_{k-1}\quad\cdots\quad y_{k-n}\quad u_{k-1}\quad\cdots\quad u_{k-n}\\&r_{\mathrm{f},k}\quad\cdots\quad r_{\mathrm{f},k-n_\mathrm{s}}],\end{aligned}\tag{37}$$

$$\Phi_{\mathrm{f},k}\triangleq N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}\Phi_k,\tag{38}$$

$$u_{\mathrm{f},k}\triangleq N_{n_\mathrm{r}}N_u(\mathbf{q}^{-1})\mathbf{q}^{-n_\mathrm{r}}u_k.\tag{39}$$

$$z_{\mathrm{f},k}\triangleq D_\mathrm{m}(\mathbf{q}^{-1})F(\mathbf{q}^{-1})(y_k-y_{\mathrm{m},k}).\tag{40}$$

Since $N_\mathrm{c}^*(\mathbf{q}^{-1})$, $D_\mathrm{c}^*(\mathbf{q}^{-1})$, and $R_\mathrm{c}^*(\mathbf{q}^{-1})$ are unknown, the goal is to solve the regression (35) at each step k to obtain the estimate $\hat{\theta}_k$. The estimation error is thus given by

$$\hat{z}_k(\hat{\theta}_k)\triangleq z_{\mathrm{f},k}-u_{\mathrm{f},k}+\Phi_{\mathrm{f},k}\hat{\theta}_k.\tag{41}$$

For regression at each step, recursive least squares (RLS) is used to minimize the cost function

$$\begin{aligned}J_k(\hat{\theta}_k)\triangleq&\sum_{i=1}^{k}\lambda^{k-i}[\hat{z}_i(\hat{\theta}_i)^\mathrm{T}\hat{z}_i(\hat{\theta}_i)]\\&+\lambda^k\left(\hat{\theta}_k-\hat{\theta}_0\right)^\mathrm{T}R_\theta\left(\hat{\theta}_k-\hat{\theta}_0\right),\end{aligned}\tag{42}$$

where $\lambda\in(0,1]$ is the forgetting factor. Using the computed RLS solution and (11), the control input at step $k+1$ is given by

$$u_{k+1}=\Phi_{k+1}\hat{\theta}_{k+1}.\tag{43}$$

Note that, it is assumed N_{n_r}, $N_u(\mathbf{q}^{-1})$, n_r, and n are known a priori.

IV. Examples

This section applies RC-MRAC to a variety of systems and commands r_k. Example 4 provides an example of how RC-MRAC can be modified to handle external disturbances. Each simulation is ran for 200 steps, where the following performance metric using the model-following error $e_k=y_k-y_{\mathrm{m},k}$ is used

$$\|e\|\triangleq\sqrt{\sum_{i=101}^{200}e_i^2}.\tag{44}$$

For the command, a square-wave signal with a period of 50 steps is used.

A. Example 1: Minimum-Phase System

Consider the system

$$\frac{N(\mathbf{q}^{-1})}{D(\mathbf{q}^{-1})}=\frac{\mathbf{q}^{-1}-0.5\mathbf{q}^{-2}}{(1-\rho e^{\jmath\nu}\mathbf{q}^{-1})(1-\rho e^{-\jmath\nu}\mathbf{q}^{-1})},\tag{45}$$

and reference model

$$\frac{N_\mathrm{m}(\mathbf{q}^{-1})}{D_\mathrm{m}(\mathbf{q}^{-1})}=\frac{\mathbf{q}^{-1}-0.2\mathbf{q}^{-2}}{(1-0.5e^{\jmath\frac{\pi}{2}}\mathbf{q}^{-1})(1-0.5e^{-\jmath\frac{\pi}{2}}\mathbf{q}^{-1})}.\tag{46}$$

We now demonstrate the model-following performance of RC-MRAC for various values of ρ and ν for square-wave commands. $F(\mathbf{q}^{-1})$ is chosen as

$$F(\mathbf{q}^{-1})=(1+0.5\mathbf{q}^{-1}),\tag{47}$$

and RLS is initialized with $\hat{\theta}_0=0_{6\times1}$, $R_\theta=10^{-5}I_6$ and $\lambda=1$.

The model-following error versus the pole locations of the system is shown in Figure 2 for various values of ρ and ν. Note that the model-following performance degrades when the system poles are closer to the system zero. The response of the system for $\rho=0.5$ and $\nu=\frac{\pi}{4}$ is given in Figure 3, and the resulting closed-loop system after 200 steps is compared to the reference model (46) in Figure 4.

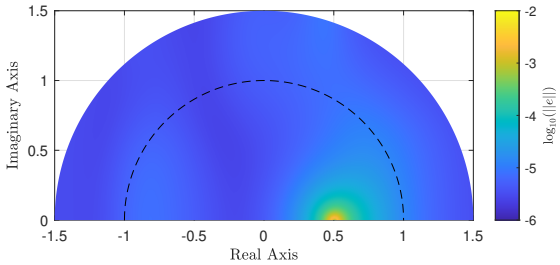

Fig. 2. Example 1. The log of the model-following error metric is shown versus the pole locations of the system.

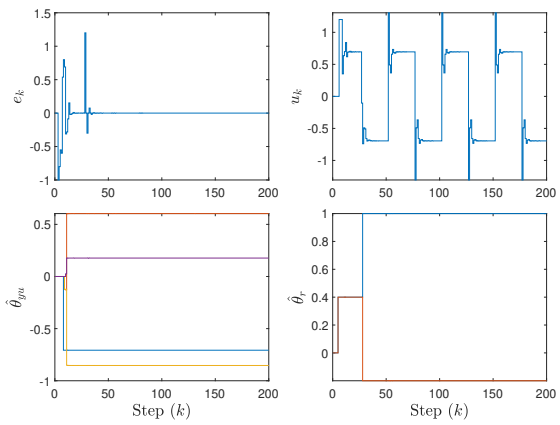

Fig. 3. Example 1. The response of the system for $\rho = 0.5$ and $\nu = \frac{\pi}{4}$ is shown. Clockwise from top left shows model-following error e_k, control input u_k, controller coefficients $\hat{\theta}$ associated with r_k, and controller coefficients $\hat{\theta}$ associated with y_k and u_k.

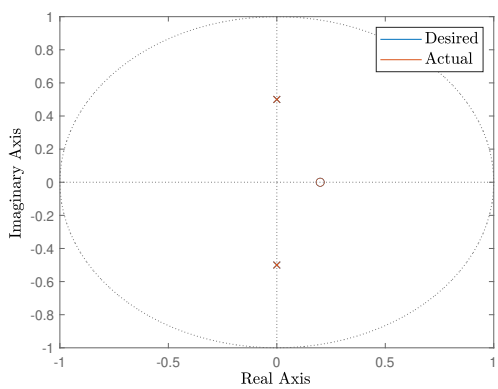

Fig. 4. Example 1. At step 200, the desired closed-loop poles and zeros are shown along with the actual closed-loop poles and zeros. Note that the actual closed-loop system matches the reference model.

B. Example 2: NMP System

Consider the system

$$\frac{N(\mathbf{q}^{-1})}{D(\mathbf{q}^{-1})} = \frac{\mathbf{q}^{-1} - 1.5\mathbf{q}^{-2}}{(1-\rho e^{\jmath\nu}\mathbf{q}^{-1})(1-\rho e^{-\jmath\nu}\mathbf{q}^{-1})}, \tag{48}$$

and reference model

$$\frac{N_{\mathrm{m}}(\mathbf{q}^{-1})}{D_{\mathrm{m}}(\mathbf{q}^{-1})} = \frac{\mathbf{q}^{-1} - 1.5\mathbf{q}^{-2}}{(1-0.5e^{\jmath\frac{\pi}{2}}\mathbf{q}^{-1})(1-0.5e^{-\jmath\frac{\pi}{2}}\mathbf{q}^{-1})}. \tag{49}$$

We now demonstrate the model-following performance of RC-MRAC on a NMP system for various values of ρ and ν for square-wave commands. $F(\mathbf{q}^{-1})$ is chosen as

$$F(\mathbf{q}^{-1}) = (1 - 0.25\mathbf{q}^{-2}), \tag{50}$$

and RLS is initialized with $\hat{\theta}_0 = 0_{5\times 1}$, $R_\theta = 10^{-5} I_5$ and $\lambda = 1$.

The model-following error versus the pole locations of the system is shown in Figure 5 for various values of ρ and ν. The response of the system for $\rho = 0.5$ and $\nu = \frac{\pi}{4}$ is given in Figure 6, and the resulting closed-loop system after 200 steps is compared to the reference model (49) in Figure 7. Note that the controller convergence rate is much slower than in the minimum-phase case leading to higher model-following error after 200 steps.

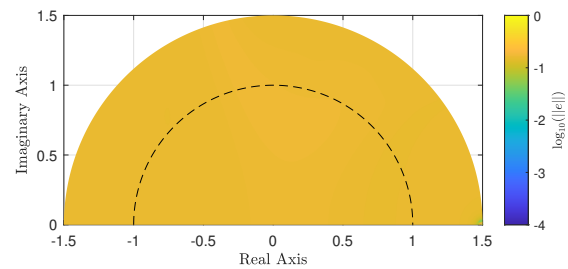

Fig. 5. Example 2. The log of the model-following error metric is shown versus the pole locations of the system.

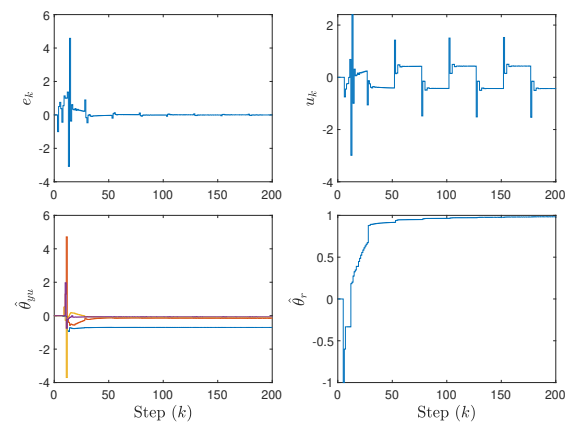

Fig. 6. Example 2. Response of the system is shown for $\rho = 0.5$ and $\nu = \frac{\pi}{4}$. Clockwise from top left shows model-following error e_k, control input u_k, controller coefficients $\hat{\theta}$ associated with r_k, and controller coefficients $\hat{\theta}$ associated with y_k and u_k.

C. Example 3: Uncertain NMP System

Consider the system in Example 2, but now the NMP zero is uncertain with true location $z = 1.5$ and assumed location $z = 1.4$. The reference model then becomes

$$\frac{N_{\mathrm{m}}(\mathbf{q}^{-1})}{D_{\mathrm{m}}(\mathbf{q}^{-1})} = \frac{\mathbf{q}^{-1} - 1.4\mathbf{q}^{-2}}{(1-0.5e^{\jmath\frac{\pi}{2}}\mathbf{q}^{-1})(1-0.5e^{-\jmath\frac{\pi}{2}}\mathbf{q}^{-1})}. \tag{51}$$

We now demonstrate the model-following performance of RC-MRAC on the uncertain NMP system for various values of ρ and ν for square-wave commands. $F(\mathbf{q}^{-1})$ and RLS are initialized as in Example 2.

The model-following error versus the pole locations of the system is shown in Figure 8 for various values of ρ and ν. Note that the system remains stable for all tested system values despite the uncertain NMP zero. The large error is due to the inability of

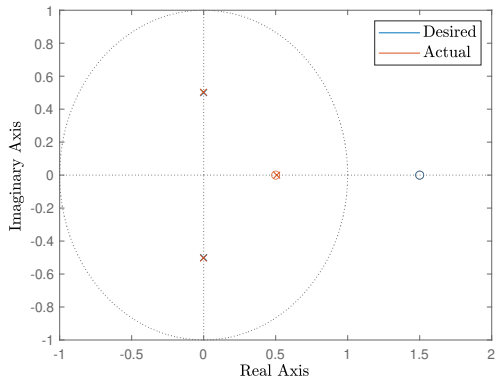

Fig. 7. Example 2. At step 200, the desired closed-loop poles and zeros are shown along with the actual closed-loop poles and zeros. Note that the actual closed-loop system is very close to the reference model.

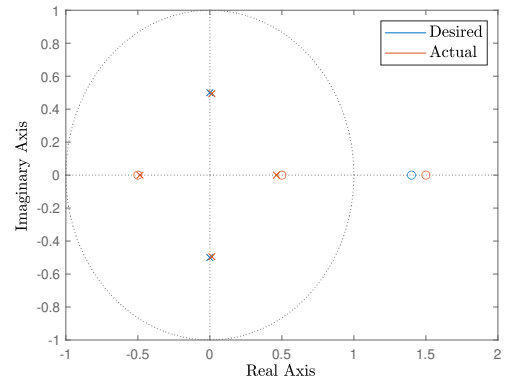

Fig. 10. Example 3. At step 200, the desired closed-loop poles and zeros are shown along with the actual closed-loop poles and zeros. Note the discrepancy between the actual unmodeled zero at $z=1.5$ and the reference model zero at $z=1.4$.

feedback control to move the NMP zero from its location $z=1.5$ to the desired location $z=1.4$. The response of the system for $\rho=0.5$ and $\nu=\frac{\pi}{4}$ is given in Figure 9, and the resulting closed-loop system after 200 steps is compared to the reference model (51) in Figure 10.

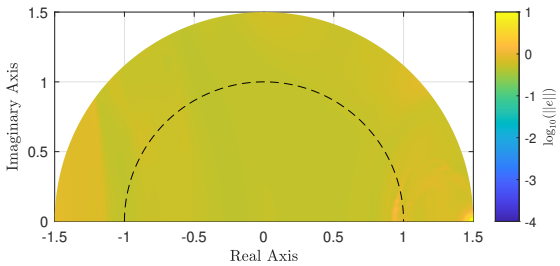

Fig. 8. Example 3. The log of the model-following error metric is shown versus the pole locations of the system.

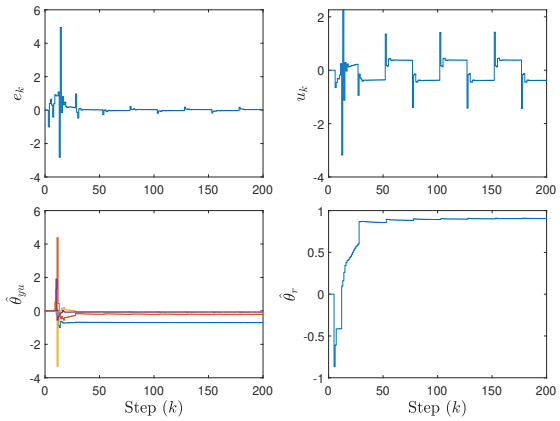

Fig. 9. Example 3. Response of the system is shown for $\rho=0.5$ and $\nu=\frac{\pi}{4}$. Clockwise from top left shows model-following error e_k, control input u_k, controller coefficients $\hat{\theta}$ associated with r_k, and controller coefficients $\hat{\theta}$ associated with y_k and u_k.

D. Example 4: Harmonic Disturbance

Consider the same system, reference model, and $F(\mathbf{q}^{-1})$ as in Example 1. We now place an unknown single harmonic disturbance at a frequency of 0.35 radians per step at the input of the system. To accomplish harmonic disturbance rejection and model-following, we increase the order used in the controller to $n=4$, and set $n_s=3$ to match the desired closed-loop relative degree. RLS is initialized with $\hat{\theta}_0=0_{12\times1}$, $R_\theta=10^{-5}I_{12}$ and $\lambda=1$

The model-following error versus the pole locations of the system is shown in Figure 11 for various values of ρ and ν. Note that the model-following performance degrades when the system poles are closer to the system zero. The response of the system for $\rho=0.5$ and $\nu=\frac{\pi}{4}$ is given in Figure 12, and the resulting closed-loop system after 200 steps is compared to the reference model (46) in Figure 13. In Figure 13 notice that the closed-loop system from the disturbance w to the measurement y has zeros at the disturbance frequency.

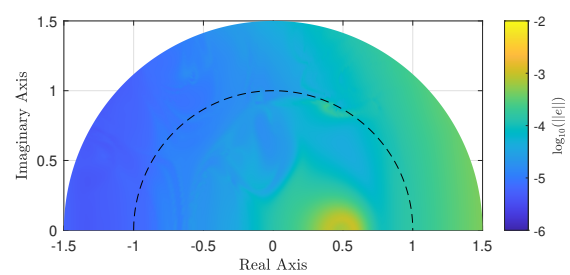

Fig. 11. Example 4. The log of the model-following error metric is shown versus the pole locations of the system.

E. Example 5: Comparison with Elliot's Adaptive Pole-Placement Controller

Consider the same system, reference model, and $F(\mathbf{q}^{-1})$ as in Example 2. To demonstrate the reduced persistency requirements of RC-MRAC, we compare the command following performance between Elliot's adaptive pole-placement controller (APPC) to RC-MRAC for a step command of height 0.01. The arbitrary polynomial $q(\mathbf{q}^{-1})$ in [9] is chosen to be equal to $F(\mathbf{q}^{-1})$. Both algorithms attempt to place the closed-loop poles at $D_{\mathrm{m}}(\mathbf{q}^{-1})$. The model-following error versus the pole locations of the system is shown in Figure 14 for various values of ρ and ν. RC-MRAC has a more consistent response and lower error compared to APPC for a wide range of plant values.

V. Conclusion

Retrospective cost model reference adaptive control (RC-MRAC) was developed and investigated. This controller places the closed-loop poles of the system to match the desired closed-loop

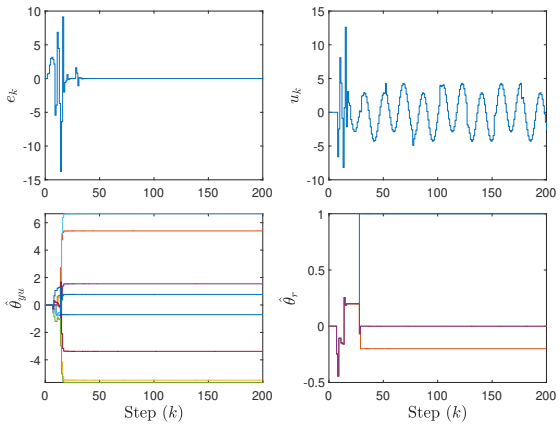

Fig. 12. Example 4. Response of the system is shown for $\rho = 0.5$ and $\nu = \frac{\pi}{4}$. Clockwise from top left shows model-following error e_k, control input u_k, controller coefficients $\hat{\theta}$ associated with r_k, and controller coefficients $\hat{\theta}$ associated with y_k and u_k.

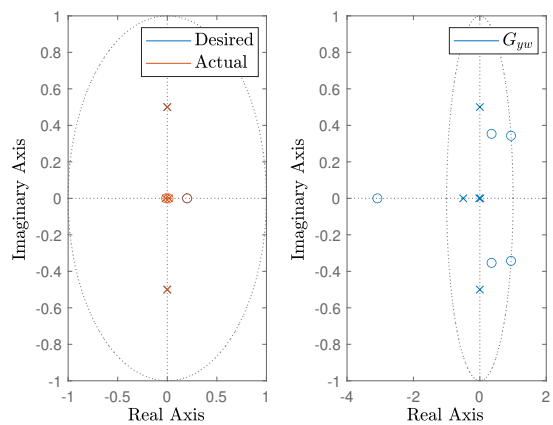

Fig. 13. Example 4. Left shows the desired closed-loop poles and zeros versus the actual closed-loop poles and zeros at step 200. Note that the actual closed-loop system is very close to the reference model. Right shows the closed-loop response between the disturbance and the measurement. Note that the controller places zeros at the disturbance frequencies

poles given by a reference model provided that the leading numerator coefficient, relative degree, system order, and NMP zeros are known. RC-MRAC was shown to handle uncertainty in the NMP zero knowledge and was stable over a wide range of systems. Additionally, it was shown that, with a slight modification, RC-MRAC can reject harmonic disturbances. For situations with little persistency, RC-MRAC outperforms Elliot's APPC at the price of knowledge of the NMP zeros.

Future work will extend RC-MRAC to the MIMO case following a similar development for RCAC given in [14]. A key challenge is the development of stability results for RC-MRAC. Given the development of stability results for similar algorithms [8], [9], a stability result for RC-MRAC will closely follow established arguments.

Acknowledgments

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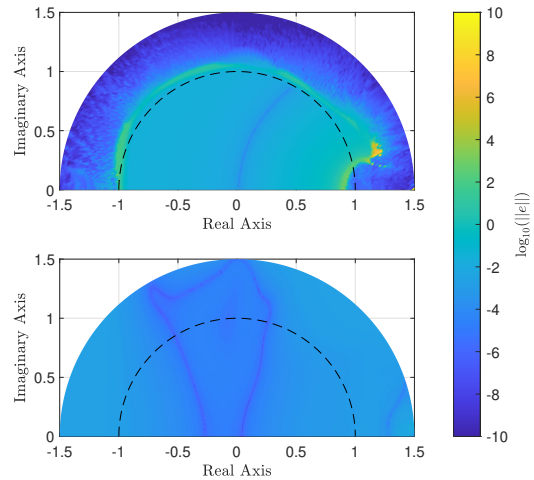

Fig. 14. Example 5. Elliot APPC Comparison. Log of the model-following error metric is shown versus the pole locations of the system. Top shows the model-following error for Elliot's APPC. Bottom shows the model-following error for RC-MRAC. Note that Elliot's APPC becomes unstable for some plant poles due to the lack of persistency.

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