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On the Dynamics of a Bead Sliding on a Freely Rotating Slanted Wire

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Abstract

We consider the dynamics of a bead that slides frictionlessly along a rotating slanted wire of infinite length and finite moment of inertia under the influence of uniform gravity. Since the wire is frictionless and external forces are absent, energy and angular momentum are conserved, and the angular velocity of the wire may increase or decrease as the bead and the wire exchange energy. Assuming that the bead is initially at rest along the wire, this paper characterizes all possible trajectories of the bead along the wire, which may be constant, divergent, oscillatory, or convergent.

Keywords Centrifugal force · Lagrangian dynamics · Periodic solution

Mathematical subject classification 34C25

1 Introduction

Fictitious forces arise when applying Newton's second law in non-inertial frames, that is, frames that rotate relative to an inertial frame (Taylor 2005, pp. 342–354; Kibble and Berkshire 2004, pp. 111–117). The interplay between real and fictitious forces is critical in many applications. For example, since gravity is attractive and centrifugal force is repulsive, the relative magnitudes of these quantities determine the resulting trajectories of the interacting bodies (Rubin 1983; Jonsson 2006). Like the Coriolis force, which is also fictitious, centrifugal force is a force-like ersatz for mass times non-inertial acceleration.

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The interaction between gravity and centrifugal force has been studied through low-dimensional examples involving a bead sliding on a rotating wire. The case of a circular hoop rotating around a vertically oriented diameter was considered in Arnold (1989, p. 87), Strogatz (1994, pp. 61–63), and Dutta and Ray (2011), while the case of a circular hoop rotating around a horizontally oriented diameter was studied in Johnson and Rabchuk (2009). A rotating parabolically shaped wire was considered in Hatimi and Ganji (2014), Baleanu et al. (2017), Moatimid (2020), and a slanted straight wire rotating around a vertical axis was studied in Baleanu et al. (2018). In all of these studies, the angular rate of the wire is assumed to be constant, and thus, each of these mechanical systems has one degree of freedom.

The present paper considers the case of a straight, frictionless wire of infinite length and finite moment of inertia free of external forces, and thus whose angular rate can increase or decrease as the bead moves frictionlessly along it. In the case of a horizontal wire, gravity plays no role, and the centrifugal force is parallel with the wire. The true (non-fictitious) force, however, is due to contact with the wire and thus is perpendicular to the wire. For this system, the second-order differential equation for the position of the bead was solved in Kouba and Bernstein (2021) in terms of Legendre's elliptic integrals. This solution provided an explicit expression for the terminal velocity of the bead along the wire as well as the asymptotic angle of the wire around the vertical axis, that is, as $t \to \infty$, the limiting angle of the wire as it rotates in the horizontal plane. The differential equation obtained in Kouba and Bernstein (2021) is not included in the compendium (Sachdev 1997).

The present paper generalizes the problem considered in Kouba and Bernstein (2021) by assuming that the wire is slanted at a fixed angle. This formulation extends the slanted wire problem considered (Baleanu et al. 2018) by allowing the angular velocity of the wire to change in response to the motion of the bead. Since the wire is frictionless and no external forces are applied to the wire, energy and angular momentum are conserved. Consequently, as in Kouba and Bernstein (2021), the constants of the motion allow the fourth-order ordinary differential equation for the dynamics of the bead and wire to be reduced to a third-order differential equation, of which the second-order dynamics of the bead along the wire can be analyzed separately from the motion of the wire around the horizontal axis. Unlike the second-order differential equation analyzed in Kouba and Bernstein (2021), however, the equation for the slanted wire does not appear to be expressible in terms of known functions.

The contribution of the present paper is a complete elucidation of the solutions of the second-order differential equation arising from the freely rotating slanted wire assuming that the initial velocity of the bead along the wire is zero. The surprising outcome of this analysis is the fact that these solutions include (1) a fixed position, (2) convergence, (3) divergence to $-\infty$, and (4) periodic oscillations. These outcomes are analyzed in terms of seven cases that are classified in terms of the initial bead position and initial wire angular velocity. The richness of the properties of this system stands in stark contrast to the case of a horizontal wire considered in Kouba and Bernstein (2021), where the bead can only diverge.

The contents of the paper are as follows. In Sect. 2 we formulate the problem and introduce notation for non-dimensionalizing the solution of the dynamical equation for the position of the bead. Section 3 focuses on Case 1, where the position of the

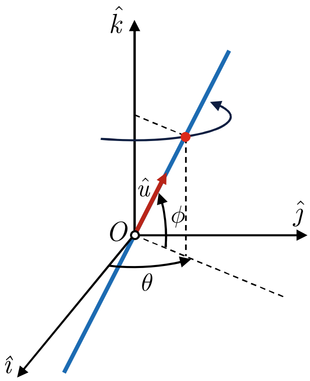

Fig. 1 A bead (red dot) sliding on a slanted wire (blue line). ϕ is the angle from the horizontal plane to the wire (Color figure online)

bead is fixed, namely $z \equiv z_0 = z_e$. Section 4 analyzes the motion of the bead in the six remaining cases, where $z_0 \neq z_e$. Stability implications of these cases are discussed in Sect. 5. Section 6 illustrates these cases with numerical examples.

2 Problem Formulation

We consider a wire passing through a fixed point O, making an angle $\phi \in (0, \pi/2)$ with the horizontal plane $(O; \hat{\imath}, \hat{\jmath})$. The wire rotates frictionlessly around the vertical axis oriented by the unit vector $\hat{k}$. Let J_0 denote the moment of inertia of the wire around $\hat{k}$ in the case where $\phi = 0$. Then, the moment of inertia of the wire when it rotates around $\hat{k}$ slanted by the constant angle ϕ is $J_0 \cos^2 \phi$. Although the wire is assumed to have infinite length, the finiteness of J_0 can be viewed as a consequence of an exponential convergence to zero of the diameter of the wire as the distance from the center of the wire increases. On the other hand, if the wire has finite length, then its length must be sufficient to support the periodic bead motion discussed in this paper. In cases where the bead position diverges to infinity, the bead will fall off the end of the wire and continue in projectile motion. The bead b with mass m slides without friction along the wire. The wire is oriented by the unit vector $\hat{u}$, and thus, at time s, the position of the bead is given by $\vec{r}_{\mathrm{b}}(s) = x(s)\hat{u}(s)$. The angle $\theta \in (-\pi, \pi]$ is the angle from $\hat{\imath}$ to the projection of $\hat{u}$ onto the $\hat{\imath}$–$\hat{\jmath}$ plane, where the direction of increasing θ is determined by the right-hand rule with the right thumb pointing in the direction of $\hat{k}$ (Fig. 1).

Note that, letting $(\dot{})$ denote $\mathrm{d}/\mathrm{d}s$,

$$\vec{r}_{\mathrm{b}} = x((\cos\phi)\cos\theta\hat{\imath} + (\cos\phi)\sin\theta\hat{\jmath} + \sin\phi\hat{k}), \tag{1}$$

$$\vec{v}_{\mathrm{b}} = (\dot{x}\cos\theta - x\dot{\theta}\sin\theta)\cos\phi\hat{\imath} + (\dot{x}\sin\theta + x\dot{\theta}\cos\theta)\cos\phi\hat{\jmath} + \dot{x}\sin\phi\hat{k}, \tag{2}$$

and thus

$$\|\vec{v}_{\mathrm{b}}\|^2 = (\cos^2\phi)[(\dot{x}\sin\theta - x\dot{\theta}\cos\theta)^2 + (\dot{x}\cos\theta + x\dot{\theta}\sin\theta)^2] + \dot{x}^2\sin^2\phi$$

$$= \dot{x}^2 + (\cos^2\phi)x^2\dot{\theta}^2. \tag{3}$$

Using (3), the kinetic energy of the system is given by

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}(mx^2\cos^2\phi + J)\dot{\theta}^2, \tag{4}$$

and thus the Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}(mx^2 + J_0)(\cos^2\phi)\dot{\theta}^2 - mg(\sin\phi)x. \tag{5}$$

Note that the Lagrangian does not depend on θ. Hence, θ is an ignorable coordinate (Greenwood 1977, p. 67), which implies the generalized momentum corresponding to θ is a constant of the motion.

Next,

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0 \tag{6}$$

implies that $(mx^2 + J_0)\dot{\theta}$ is constant. Defining $\omega_0 \triangleq \dot{\theta}_0 \triangleq \dot{\theta}(0)$, $x_0 \triangleq x(0)$, and

$$c \triangleq (mx_0^2 + J_0)\omega_0, \tag{7}$$

it follows that

$$\dot{\theta} = \frac{c}{mx^2 + J_0} = \omega_0\frac{mx_0^2 + J_0}{mx^2 + J_0}, \tag{8}$$

which expresses conservation of angular momentum. Similarly,

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = 0 \tag{9}$$

yields

$$\ddot{x} = (\cos^2\phi)\dot{\theta}^2 x - g\sin\phi. \tag{10}$$

Substituting $\dot{\theta}$ from (8) into (10) yields

$$\ddot{x} = c^2\cos^2\phi\,\frac{x}{(mx^2 + J_0)^2} - g\sin\phi. \tag{11}$$

Next, define

$$\lambda \triangleq \sqrt{\frac{m}{J_0}}, \quad \mu \triangleq \frac{J_0}{c\cos\phi}, \tag{12}$$

the dimensionless time variable

$$t \triangleq \frac{s}{\mu}, \tag{13}$$

and the dimensionless bead position

$$z(t) \triangleq \lambda x(\mu t) = \lambda x(s). \tag{14}$$

Henceforth, letting $\dot{(\ )}$ denote dimensionless $\mathrm{d}/\mathrm{d}t$, (11) implies

$$\begin{aligned}\ddot{z}(t) &= \lambda\mu^2\ddot{x}(\mu t)\\ &= \lambda\mu^2 c^2\cos^2\phi\,\frac{x(\mu t)}{[mx^2(\mu t)+J_0]^2} - \lambda\mu^2 g\sin\phi\\ &= \mu^2\lambda^4 c^2\cos^2\phi\,\frac{z(t)}{[mz(t)^2+J_0\lambda^2]^2} - \lambda\mu^2 g\sin\phi\\ &= \frac{z(t)}{[z(t)^2+1]^2} - \frac{J_0\sqrt{mJ_0}}{c^2\cos^2\phi}g\sin\phi.\end{aligned} \tag{15}$$

Hence, (15) can be written as

$$\ddot{z}(t) = \frac{z(t)}{[z(t)^2+1]^2} - a, \tag{16}$$

where the dimensionless constant $a > 0$ is defined by

$$a \overset{\triangle}{=} \frac{gJ_0\sqrt{mJ_0}}{c^2}\frac{\sin\phi}{\cos^2\phi} = g\lambda\mu^2\sin\phi. \tag{17}$$

Furthermore, note that

$$c = J_0\omega_0(z_0^2+1), \tag{18}$$

where $z_0 \overset{\triangle}{=} z(0) = \lambda x_0$.

Next, we define $\eta > 0$ by

$$\eta \overset{\triangle}{=} \frac{1}{g}\sqrt{\frac{J_0}{m}}\frac{\cos^2\phi}{\sin\phi} = \frac{1}{\lambda g}\frac{\cos^2\phi}{\sin\phi}, \tag{19}$$

which is independent of the initial conditions, and the constant $z_\mathrm{e} > 0$ by

$$z_\mathrm{e} \overset{\triangle}{=} \frac{1}{\eta\omega_0^2}. \tag{20}$$

With this notation, a can be written as

$$a = \frac{1}{\eta\omega_0^2(z_0^2+1)^2} = \frac{z_\mathrm{e}}{(z_0^2+1)^2}. \tag{21}$$

Note that a depends on both z_0 and ω_0. Moreover, setting $t = 0$ in (16) yields

$$\ddot{z}_0 \overset{\triangle}{=} \ddot{z}(0) = \frac{z_0 - z_\mathrm{e}}{(z_0^2+1)^2}. \tag{22}$$

Since the vector field corresponding to (16) is C^1, it follows that, for all $(t_0, z_0, \dot{z}_0)$, there exists a unique maximal C^1 solution $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(t_0) = z_0$ and

$\dot{\varphi}(t_0) = \dot{z}_0$. We consider only the case $t_0 = 0$ and $\dot{z}_0 = 0$. Additional solutions exist by considering nonzero $\dot{z}_0$. For example, if $\dot{z}_0 \neq 0$ and the wire is initially not rotating, then, for all $t > 0$, the wire does not rotate, and the bead eventually diverges to $-\infty$. Nevertheless, the case $\dot{z}_0 = 0$ suffices to illustrate the complexity inherent in this system.

Definition 1 Given $(z_0, \omega_0) \in \mathbb{R} \times \mathbb{R}\backslash\{0\}$, let $Z_{z_0,\omega_0} : \mathbb{R} \to \mathbb{R}$ be the unique maximal solution of the ordinary differential equation

$$\ddot{z} = \frac{z}{(z^2+1)^2} - a \tag{23}$$

that satisfies the initial conditions $Z_{z_0,\omega_0}(0) = z_0$ and $\dot{Z}_{z_0,\omega_0}(0) = 0$, where a is given by (21).

Note that (23) is a non-homogeneous ordinary differential equation due to the constant input $-a$. An unusual feature of (23) is the fact that a depends on the initial condition z_0 as well as ω_0. The following sections describe and characterize in terms of z_0 and ω_0 the *unique* solution $z \triangleq Z_{z_0,\omega_0}$ of (23).

Note that (23) implies that, for all $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{z}^2 + \frac{1}{z^2+1} + 2az\right) = 0. \tag{24}$$

Hence, $\dot{z}^2 + \frac{1}{z^2+1} + 2az$ is a constant of the motion. To show that this constant is a consequence of conservation of energy, note that the total system energy is given by

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(mx^2 + J_0)(\cos^2\phi)\dot{\theta}^2 + mg(\sin\phi)x \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{c^2\cos^2\phi}{mx^2+J_0} + mg(\sin\phi)x \\
&= \frac{m}{2\lambda^2\mu^2}\dot{z}^2 + \frac{1}{2}\frac{c^2\cos^2\phi}{(m/\lambda^2)z^2+J_0} + \frac{mg(\sin\phi)}{\lambda}z \\
&= \frac{c^2\cos^2\phi}{2J_0}\dot{z}^2 + \frac{c^2(\cos^2\phi)}{2J_0}\frac{1}{1+z^2} + \frac{mg(\sin\phi)}{\lambda}z \\
&= \frac{c^2\cos^2\phi}{2J_0}\left(\dot{z}^2 + \frac{1}{1+z^2} + 2az\right).
\end{aligned} \tag{25}$$

Finally, to analyze the motion of the wire, we define

$$\Theta(t) \triangleq \frac{1}{\mu}\theta(\mu t) \tag{26}$$

and note that (8) implies

$$\dot{\Theta}(t) = \dot{\theta}(\mu t) = \omega_0\frac{z_0^2+1}{z(t)^2+1}. \tag{27}$$

Without loss of generality, we assume throughout the paper that $\theta(0) = 0$.

The following sections show that the bead has four types of motion along the wire, namely fixed position, convergent, divergent, and periodic. In the enumeration of these various cases, the focus is on the motion of the bead given by $z(t)$. The corresponding motion of the angle of the wire around the vertical support follows directly from (27).

3 Case 1: The Solution with $z_0 = z_e$

In this section, we consider Case 1, which assumes that $z_0 = z_e$. We show that, under this assumption, the solution $z = Z_{z_0,\omega_0}$ (see Definition 1) is constant, and thus, the bead has a fixed location along the wire, which depends on the initial angular rate of the wire.

Proposition 2 **(Case 1)** *The following statements are equivalent:*

(i) $z_0 = z_e$.
(ii) $\ddot{z}_0 = 0$.
(iii) $z(t) = z_e$ *for all* $t \in \mathbb{R}$.

Proof The equivalence (i) $\Leftrightarrow$ (ii) follows from (22), and (iii) $\Rightarrow$ (i) is immediate. To prove the converse, we note, by the uniqueness statement in Definition 1, it suffices to show that the constant function $\varphi(t) \equiv z_e$ satisfies (23) and the initial conditions. Using (21) and (i), it follows that (23) can be written as

$$\ddot{z} = \frac{z}{(z^2+1)^2} - \frac{z_e}{(z_e^2+1)^2},$$

which proves (i) $\Rightarrow$ (iii). □

Under the assumptions of Proposition 2, the bead has the fixed location along the wire given by

$$x_e \triangleq \frac{z_e}{\lambda} = \frac{g \sin\phi}{\omega_0 \cos^2\phi}. \tag{28}$$

The bead thus moves with constant speed along a circle in three-dimensional space with constant angular rate $\dot{\Theta}(0) = \dot{\theta}(0) \equiv \omega_0$ according to (27).

Proposition 2 shows that, if $z_0 = z_e$, then the solution (assuming $\dot{z}_0 = 0$) of (23) is $z(t) \equiv z_e$. Hence, for the remaining cases, we assume that $z_0 \neq z_e$.

4 Case 2: The Solution with $z_0 \neq z_e$

In this section, we describe the solution $z = Z_{z_0,\omega_0}$ (see Definition 1) under the assumption $z_0 \neq z_e$, which is equivalent to $\ddot{z}_0 \neq 0$ according to Proposition 2. Under this assumption, we show that the position of the bead along the wire may diverge, converge, or oscillate. The following lemma is needed.

Lemma 3 *For all* $t \in \mathbb{R}$,

$$\dot{z}(t)^2 = -\frac{2a}{z(t)^2+1}(z(t)-z_0)Q(z(t)), \tag{29}$$

where Q *is the quadratic polynomial*

$$Q(X) \triangleq X^2 - \frac{1}{2a(z_0^2+1)}X + 1 - \frac{z_0}{2a(z_0^2+1)}. \tag{30}$$

Furthermore, the discriminant Δ *of* Q *is given by*

$$\Delta = \frac{4}{z_{\mathrm{e}}^2}\left(g(-z_0)+z_e\right)\left(g(z_0)-z_e\right), \tag{31}$$

where

$$g(z_0) \triangleq \frac{\left(\sqrt{1+z_0^2}+z_0\right)(1+z_0^2)}{4} = \frac{1+z_0^2}{4\left(\sqrt{z_0^2+1}-z_0\right)}. \tag{32}$$

Proof Since $\dot{z}_0 = 0$, (24) implies that, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\dot{z}^2 &= \frac{1}{z_0^2+1} - \frac{1}{z^2+1} + 2a(z_0 - z) \\
&= -2a\frac{z-z_0}{z^2+1}\left[z^2+1-\frac{z+z_0}{2a(z_0^2+1)}\right] \\
&= -\frac{2a}{z^2+1}(z-z_0)Q(z).
\end{aligned}$$

The discriminant Δ of Q is thus given by

$$\begin{aligned}
\Delta &= \frac{1}{4a^2(z_0^2+1)^2} + \frac{2z_0}{a(z_0^2+1)} - 4 \\
&= \frac{(1+z_0^2)^2}{4z_{\mathrm{e}}^2} + \frac{2z_0(1+z_0^2)}{z_{\mathrm{e}}} - 4 \\
&= \frac{4}{z_{\mathrm{e}}^2}\left[\frac{(1+z_0^2)^2}{16} + \frac{1}{2}z_0(1+z_0^2)z_{\mathrm{e}} - z_{\mathrm{e}}^2\right] \\
&= \frac{4}{z_{\mathrm{e}}^2}\left[\frac{(1+z_0^2)^3}{16} - \left(z_{\mathrm{e}} - \frac{z_0(1+z_0^2)}{4}\right)^2\right]
\end{aligned}$$

$$= \frac{4}{z_e^2}\left[\frac{\left(\sqrt{1+z_0^2}-z_0\right)(1+z_0^2)}{4}+z_e\right]\left[\frac{\left(\sqrt{1+z_0^2}+z_0\right)(1+z_0^2)}{4}-z_e\right]$$
$$= \frac{4}{z_e^2}\left(g(-z_0)+z_e\right)\left(g(z_0)-z_e\right).$$

□

It will be helpful to note that

$$\operatorname{sgn}(\Delta) = \operatorname{sgn}[g(z_0)-z_e]. \tag{33}$$

Furthermore,

$$Q(z_0) = -\frac{z_0^2+1}{a}\left[\frac{z_0}{(z_0^2+1)^2}-a\right] = -\frac{z_0^2+1}{a}\ddot{z}_0$$
$$= (z_0^2+1)(1-z_0\eta\omega_0^2) = \frac{(z_0^2+1)(z_e-z_0)}{z_e}, \tag{34}$$

which implies that $Q(z_0)=0$ if and only if $z_0=z_e$ and that

$$\operatorname{sgn}(Q(z_0)) = \operatorname{sgn}(z_e-z_0). \tag{35}$$

In addition, note that (29) implies that, for all $t\in\mathbb{R}$, the solution z of (23) satisfies

$$(z(t)-z_0)Q(z(t)) \le 0. \tag{36}$$

We consider three subcases of Case 2 that depend on the sign of Δ.
Case 2.1: $\Delta<0$. The next lemma characterizes this case in terms of z_0 and ω_0.

Lemma 4 *The following conditions are equivalent:*

(i) $z_0 \neq z_e$ *and* $\Delta<0$.
(ii) $z_0 < z_e$ *and* $\Delta<0$.
(iii) $g(z_0) < z_e$.

Proof (ii) $\Rightarrow$ (i) is immediate, and (i) $\Rightarrow$ (iii) follows from (33). Finally, (iii) implies $\Delta<0$, and thus, $\operatorname{sgn}(Q)$ is constant. Since $Q(z_0)>0$, it follows from (35) that $z_0<z_e$. □

Proposition 5 **(Case 2.1)** *Assume that* $g(z_0)<z_e$. *Then, for all* $t\in\mathbb{R}$, $z(t)\le z_0$. *Furthermore, as* $t\to\infty$,

$$z(t)\sim -\tfrac{a}{2}t^2, \tag{37}$$

and thus $\lim_{t\to\infty} z(t) = -\infty$. *In addition, as* $t\to\infty$,

$$\Theta(t)-\Theta_\infty \sim -\frac{4\omega_0(z_0^2+1)}{3a^2t^3}, \tag{38}$$

where

$$\Theta_\infty \stackrel{\triangle}{=} \lim_{t\to\infty} \Theta(t) = \omega_0(z_0^2+1)\int_0^\infty \frac{1}{z(\tau)^2+1}\,\mathrm{d}\tau. \tag{39}$$

Proof According to Lemma 4, $\Delta < 0$. It follows that $\operatorname{sgn}(Q)$ is constant on $\mathbb{R}$. Since Q is monic, it follows that Q is positive on $\mathbb{R}$, and thus, for all $t \in \mathbb{R}$, $Q(z(t)) > 0$. It thus follows from (36) that, for all $t \in \mathbb{R}$,

$$z(t) - z_0 \le 0, \tag{40}$$

and thus, for all $t \in \mathbb{R}$, $z(t) \le z_0$.

Lemma 4(ii) and (22) imply that $\ddot{z}_0 < 0$. Since $\dot{z}_0 = 0$, it follows that there exists $\varepsilon > 0$ such that, for all $t \in [0, \varepsilon)$, $\dot{z}(t) < 0$ and thus $z(t) < z_0$.

Next, to prove that, for all $t > 0$, $\dot{z}(t) < 0$, suppose that, on $(0,\infty)$, $\dot{z}$ either changes sign or vanishes. Since, for all $t \in \mathbb{R}$, $Q(z(t)) > 0$, it follows from (29) that there exists $\hat{t} > 0$ such that $z(\hat{t}) = z_0$. Now, define

$$t_1 \stackrel{\triangle}{=} \sup\{t > 0\colon \text{for all } \tau \in (0,t)\colon z(\tau) < z_0\}, \tag{41}$$

and suppose that t_1 is finite. Since z is continuous, it follows that $z(t_1) = z_0$. In addition, (29) implies that, for all $\tau \in (0, t_1)$, $\dot{z}(\tau) \ne 0$. Since, for all $t \in [0,\varepsilon)$, $\dot{z}(t) < 0$, it follows that, for all $\tau \in (0,t_1)$, $\dot{z}(\tau) < 0$. Since z is decreasing on $(0, t_1)$, it follows that $z(t_1) < z_0$, which is a contradiction. Therefore, $t_1 = \infty$, which implies that, for all $t \in (0,\infty)$, $\dot{z}(t) < 0$. Therefore, z is decreasing on $(0,\infty)$.

Next, since z is decreasing on $[0,\infty)$, it follows from (29) that, for all $t \ge 0$,

$$-\frac{\dot{z}(t)}{2\sqrt{z_0 - z(t)}} = \frac{1}{2}\sqrt{\frac{2aQ(z(t))}{z(t)^2+1}}. \tag{42}$$

Note that, for all $t \ge 0$, the right side of (42) is positive and that $\lim_{z\to\pm\infty} \frac{1}{2}\sqrt{\frac{2aQ(z)}{z^2+1}} = \sqrt{\frac{a}{2}}$. Hence, there exists $A > 0$ such that, for all $t \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{z_0 - z(t)} = -\frac{\dot{z}(t)}{2\sqrt{z_0 - z(t)}} \ge A, \tag{43}$$

and thus, for all $t \ge 0$,

$$\sqrt{z_0 - z(t)} \ge At. \tag{44}$$

Hence, for all $t \ge 0$, $z(t) \le z_0 - A^2t^2$, and thus, $\lim\limits_{t\to\infty} z(t) = -\infty$.

Next, combining (30) and (42) implies that, for all $t \ge 0$,

$$-\frac{\dot{z}(t)}{2\sqrt{z_0 - z(t)}} = \sqrt{\frac{a}{2} - \frac{z(t)+z_0}{4(z_0^2+1)(z(t)^2+1)}}. \tag{45}$$

Let $\varepsilon > 0$. Since $\lim_{z\to-\infty} \frac{z+z_0}{4(z_0^2+1)(z^2+1)} = 0$, it follows that there exists $t_0 > 0$ such that, for all $t > t_0$,

$$0 \leq -\frac{z+z_0}{4(z_0^2+1)(z^2+1)} \leq \varepsilon. \tag{46}$$

Thus, for all $t > t_0$,

$$\sqrt{\frac{a}{2}} \leq -\frac{\dot{z}(t)}{2\sqrt{z_0 - z(t)}} \leq \sqrt{\frac{a}{2}+\varepsilon}. \tag{47}$$

Integrating (47) yields

$$\sqrt{\frac{a}{2}}(t-t_0) \leq \sqrt{z_0 - z(t)} - \sqrt{z_0 - z(t_0)} \leq \sqrt{\frac{a}{2}+\varepsilon(t-t_0)}, \tag{48}$$

and thus

$$\frac{a}{2} \leq \liminf_{t\to\infty} \frac{-z(t)}{t^2} \leq \limsup_{t\to\infty} \frac{-z(t)}{t^2} \leq \frac{a}{2} + \varepsilon. \tag{49}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{t\to\infty} \frac{-z(t)}{t^2} = \frac{a}{2}. \tag{50}$$

Hence, as $t \to \infty$, $z(t) \sim -\frac{a}{2}t^2$.

Next, to determine the asymptotic motion of the wire, note that (27) implies

$$\Theta(t) = \omega_0(z_0^2+1)\int_0^t \frac{1}{z(\tau)^2+1}\,\mathrm{d}\tau. \tag{51}$$

Since, as $t \to \infty$, $z(t) \sim -\frac{a}{2}t^2$, it follows that the integral in (51) converges to a finite value, and thus, (39) holds. Moreover, as $t \to \infty$,

$$\begin{aligned}\Theta_\infty - \Theta(t) &= \omega_0(z_0^2+1)\int_t^\infty \frac{1}{z(\tau)^2+1}\,\mathrm{d}\tau \\ &\sim \frac{4\omega_0(z_0^2+1)}{a^2}\int_t^\infty \frac{1}{\tau^4}\,\mathrm{d}\tau = \frac{4\omega_0(z_0^2+1)}{3a^2t^3},\end{aligned}$$

which implies (38). □

Results on z and Θ can be recast in terms of the original variables x and θ. We do this in the present case and leave the details to the interested reader in the remaining cases. Indeed, it follows from (37) that, as $s \to \infty$,

$$x(s) \sim -\frac{g\sin\phi}{2}s^2, \tag{52}$$

and from (38) that

$$\theta(s) - \theta_\infty \sim -\frac{4\omega_0(\lambda^2x_0^2+1)}{3\cos^4\phi\, s^3}, \tag{53}$$

where, using (39),

$$\theta_\infty \triangleq \lim_{s\to\infty} \theta(s) = \omega_0(\lambda^2 x_0^2 + 1) \int_0^\infty \frac{1}{\lambda^2 x(\sigma)^2 + 1}\, \mathrm{d}\sigma. \tag{54}$$

Case 2.2: $\Delta > 0$. This case is characterized by $z_0 \neq z_\mathrm{e}$ and $z_\mathrm{e} < g(z_0)$.

The discriminant Δ of Q is positive, and Q has distinct real roots $r_1 < r_2$ given by

$$r_1 = \tfrac{1}{4}\eta\omega_0^2(z_0^2+1) - \sqrt{\left[\tfrac{1}{4}\eta\omega_0^2(z_0^2+1) + z_0\right]^2 - (z_0^2+1)}, \tag{55}$$

$$r_2 = \tfrac{1}{4}\eta\omega_0^2(z_0^2+1) + \sqrt{\left[\tfrac{1}{4}\eta\omega_0^2(z_0^2+1) + z_0\right]^2 - (z_0^2+1)}. \tag{56}$$

Since $Q(u) = (u - r_1)(u - r_2)$, it follows that, for all $u \in (-\infty, r_1) \cup (r_2, \infty)$, $Q(u) > 0$, whereas, for all $u \in (r_1, r_2)$, $Q(u) < 0$.

Lemma 6 *Assume that $z_0 \neq z_\mathrm{e}$ and $z_\mathrm{e} < g(z_0)$. Then, $z_0 \notin \{r_1, r_2\}$, and*

$$\dot z^2 = -\frac{2a}{z^2+1}(z - z_0)(z - r_1)(z - r_2). \tag{57}$$

Proof Since $z_0 \neq z_\mathrm{e}$, Proposition 2 implies that $\ddot z_0 \neq 0$. Hence, (34) implies that $Q(z_0) \neq 0$, and thus $z_0 \notin \{r_1, r_2\}$. Finally, substituting $Q(z) = (z - r_1)(z - r_2)$ into (29) yields (57). □

Lemma 6 implies that, under the assumption that $z_0 \neq z_\mathrm{e}$ and $z_\mathrm{e} < g(z_0)$, there are three subcases of Case 2.2, namely, $z_0 < r_1$, $r_2 < z_0$, and $r_1 < z_0 < r_2$. We now consider these subcases.

Lemma 7 *Assume that $z_0 \neq z_\mathrm{e}$ and $z_\mathrm{e} < g(z_0)$. Then,*

(i) *$z_0 < r_1$ if and only if $z_0 < \min\left(z_\mathrm{e}, \frac{1}{\sqrt{3}}\right)$.*
(ii) *$z_0 > r_2$ if and only if $\frac{1}{\sqrt{3}} < z_0 < z_\mathrm{e}$.*
(iii) *$r_1 < z_0 < r_2$ if and only if $z_\mathrm{e} < z_0$.*

Moreover, if $z_\mathrm{e} < z_0$ then $z_\mathrm{e} < g(z_0)$.

Proof (i) If $z_0 < r_1$, then

$$Q(z_0) > 0 \quad \text{and} \quad z_0 < \tfrac{1}{2}(r_1 + r_2) = \tfrac{1}{4}\eta(z_0^2+1) = \frac{z_0^2+1}{4z_\mathrm{e}}. \tag{58}$$

Because of (34), the first condition is equivalent to $z_0 < z_\mathrm{e}$, and the second condition can be written as

$$4z_0 z_\mathrm{e} < z_0^2 + 1. \tag{59}$$

So, if $z_0 > 0$, then $z_0 < z_\mathrm{e} < \frac{z_0^2+1}{4z_0}$ and consequently $z_0 < 1/\sqrt{3}$. Hence, $z_0 < \min(z_\mathrm{e}, 1/\sqrt{3})$.

Conversely, assume that $z_0 < \min(z_e, 1/\sqrt{3})$. Then $z_0 < z_e$ implies that $Q(z_0) > 0$, so $z_0 \in (-\infty, r_1) \cup (r_2, +\infty)$. Moreover, since $z_0 < 1/\sqrt{3}$ then, according to (92), we have $4z_0 g(z_0) \leq 1 + z_0^2$. Now, because $z_e < g(z_0)$,

$$z_0 \leq \frac{1+z_0^2}{4g(z_0)} < \frac{1+z_0^2}{4z_e} \tfrac{1}{2}(r_1+r_2) < r_2.$$

Consequently, $z_0 < r_1$.

(ii) If $z_0 > r_2$ then

$$Q(z_0) > 0 \quad \text{and} \quad z_0 > \tfrac{1}{2}(r_1+r_2) = \frac{z_0^2+1}{4z_e}. \tag{60}$$

Because of (34), the first condition is equivalent to $z_0 < z_e$, and the second condition can be written as

$$4z_0 z_e > z_0^2 + 1. \tag{61}$$

So, $z_0 > 0$ and $z_e > (1+z_0^2)/(4z_0)$, so $g(z_0) > (1+z_0^2)/(4z_0)$ and $z_0 > 0$. Now, (92) yields $z_0 > 1/\sqrt{3}$. Hence, $\frac{1}{\sqrt{3}} < z_0 < z_e$.

Conversely, assume that $\frac{1}{\sqrt{3}} < z_0 < z_e$. Then $z_0 < z_e$ implies that $Q(z_0) > 0$, so $z_0 \in (-\infty, r_1) \cup (r_2, +\infty)$. On the other hand, the second-degree polynomial $q(X) = X^2 - 4Xz_e + 1$ satisfies $q(\frac{1}{\sqrt{3}}) = \frac{4}{\sqrt{3}}(\frac{1}{\sqrt{3}} - z_e) < 0$, and $q(z_e) = 1 - 3z_e^2 < 0$; hence, $q(z_0) < 0$ because $\frac{1}{\sqrt{3}} < z_0 < z_e$. This implies that $z_0 > \frac{1+z_0^2}{4z_e} = \frac{1}{2}(r_1+r_2) > r_1$. It follows that $z_0 > r_2$ as desired.

(iii) Note that $r_1 < z_0 < r_2$ if and only if $Q(z_0) < 0$, and (34) is equivalent to $z_e < z_0$. Moreover, if $z_0 > z_e(> 0)$, then using (91), we get

$$g(z_0) \geq z_0 > z_e, \tag{62}$$

and the proof of Lemma 7 is complete. □

Case 2.2.1: $\Delta > 0$ and $z_0 < r_1$

Proposition 8 (Case 2.2.1) *Assume that $z_0 < \min(z_e, 1/\sqrt{3})$ and $z_e < g(z_0)$. Then, $\ddot{z}_0 < 0$ and, for all $t \in \mathbb{R}$, $z(t) \leq z_0$. Furthermore, as $t \to \infty$, (37) is satisfied and $\lim_{t\to\infty} z(t) = -\infty$. In addition, as $t \to \infty$, (38) is satisfied, where Θ_∞ is given by (39).*

Proof Lemma 7(i) implies that $z_0 < r_1$. Hence, (36) yields that, for all $t \in \mathbb{R}$, $z(t)$ belongs to $(-\infty, z_0] \cup [r_1, r_2]$. Since these intervals are disjoint, $z(0) = z_0$, and z is continuous, it follows that, for all $t \in \mathbb{R}$, $z(t) \leq z_0$. Hence, for all $t \in \mathbb{R}$, $Q(z(t)) > 0$. Next, the same arguments used in the proof of Proposition 5 imply that z is decreasing on $(0, \infty)$ and the two subsequent statements hold. □

Case 2.2.2: $\Delta > 0$ and $r_2 < z_0$

Proposition 9 **(Case 2.2.2)** *Assume that* $\frac{1}{\sqrt{3}} < z_0 < z_\mathrm{e}$ *and* $z_\mathrm{e} < g(z_0)$*. Define*

$$\tau \triangleq \int_{r_2}^{z_0} \sqrt{\frac{1+u^2}{2aQ(u)(z_0-u)}}\,\mathrm{d}u. \tag{63}$$

Then, τ is finite and positive, and the following statements hold:

(i) *z is decreasing on $[0, \tau]$.*
(ii) $z(\tau) = r_2$.
(iii) *z is increasing on $[\tau, 2\tau]$.*
(iv) $z(2\tau) = z_0$.
(v) *z is even and 2τ-periodic.*

Proof Lemma 7(ii) implies that $r_2 < z_0$. Hence, (36) implies that, for all $t \in \mathbb{R}$, $z(t) \in (-\infty, r_1] \cup [r_2, z_0]$. Since z is continuous and $z(0) = z_0 \in [r_2, z_0]$, it follows that, for all $t \in \mathbb{R}$, $z(t) \in [r_2, z_0]$.

Next, since $Q(z_0) > 0$, it follows from (34) that $\ddot{z}_0 < 0$, which implies that there exists $\varepsilon > 0$ such that $\dot{z}$ is decreasing on $(0, \varepsilon)$. Since $\dot{z}_0 = 0$, it follows that $\dot{z}$ is negative on $(0, \varepsilon)$, and thus, z is decreasing on $(0, \varepsilon)$. Now, define

$$t_1 \triangleq \sup\{t > 0\colon z \text{ is decreasing on } (0, t)\}, \tag{64}$$

and note that $t_1 \geq \varepsilon$. Since, for all $t \in (0, t_1)$, $\dot{z}(t) < 0$, it follows from (29) that, for all $t \in (0, t_1)$,

$$\dot{z}(t) = -\sqrt{\frac{2a(z(t)-r_1)}{z(t)^2+1}} \cdot \sqrt{(z_0 - z(t))(z(t) - r_2)}. \tag{65}$$

Hence, for all $t \in (0, t_1)$,

$$t = \int_{z(t)}^{z_0} \sqrt{\frac{1+u^2}{2a(u-r_1)(z_0-u)(u-r_2)}}\,\mathrm{d}u = \int_{z(t)}^{z_0} \sqrt{\frac{1+u^2}{2aQ(u)(z_0-u)}}\,\mathrm{d}u. \tag{66}$$

Now, note that the integral in (63) is finite due to the fact that the singularities at r_2 and z_0 are integrable. Since $r_2 \leq z(t) \leq z_0$, it follows that the integrals in (66) are finite. Hence, for all $t \in (0, t_1)$, it follows that $t \leq \tau$, and thus $t_1 \leq \tau$.

Next, suppose that $t_1 < \tau$, which is equivalent to $z(t_1) > r_2$. Hence, (65) implies that $\dot{z}(t_1) < 0$, and thus, there exists $\delta > 0$ such that z is decreasing on $(t_1, t_1 + \delta)$. However, this contradicts the definition of t_1. Therefore, z decreases from z_0 to r_2 on $[0, \tau]$.

If $\ddot{z}(t_1) < 0$, then there exists $\delta > 0$ such that $\dot{z}$ is decreasing on $(t_1, t_1 + \delta)$. Since $\dot{z}(\tau) = 0$, it follows that $\dot{z}$ is negative on $(t_1, t_1 + \delta)$, and thus, z is decreasing on $(\tau, \tau + \delta)$. Hence, there exists $t_2 \in (0, \delta)$ such that $z(t_2) < z(t_1) = r_2$, which is a contradiction.

Now, suppose that $\ddot{z}(\tau) = 0$. It thus follows that the constant function $\tilde{z}(t) \equiv r_2$ is a solution of (23) that satisfies $\tilde{z}(\tau) = r_2$ and $\dot{\tilde{z}}(\tau) = 0$. By uniqueness, it follows that $z = \tilde{z}$ and thus, in particular, $z_0 = z(0) = \tilde{z}(0) = r_2$, which is a contradiction. It thus follows that $\ddot{z}(\tau) > 0$.

Next, define $w\colon \mathbb{R} \to \mathbb{R}$ by $w(t) = z(2\tau - t)$. Direct substitution shows that w is a solution of (23). Furthermore, $w(\tau) = z(\tau) = r_2$ and $\dot{w}(\tau) = -\dot{z}(\tau) = 0$. Uniqueness implies that $w = z$. That is $z(t) = z(2\tau - t)$ for all $t \in [\tau, 2\tau]$. Hence, z is increasing from r_2 to z_0 on $[\tau, 2\tau]$. Moreover, $z(2\tau) = z(0) = z_0$ and $\dot{z}(2\tau) = -\dot{z}_0 = 0$, which implies, again by uniqueness, that, for all t, $z(t) = z(2\tau + t)$. Therefore, z is 2τ-periodic. Finally, since $z(t) = z(2\tau - t) = z(-t)$, the solution z is an even function. □

Case 2.2.3: $\Delta > 0$ **and** $r_1 < z_0 < r_2$

Proposition 10 (Case 2.2.3) *Assume that* $z_e < z_0$, *and define*

$$\tau \triangleq \int_{z_0}^{r_2} \sqrt{\frac{1 + u^2}{2aQ(u)(z_0 - u)}}\,\mathrm{d}u. \tag{67}$$

Then, τ *is finite and positive, and the following statements hold:*

(i) z *is increasing on* $[0, \tau]$.
(ii) $z(\tau) = r_2$.
(iii) z *is decreasing on* $[\tau, 2\tau]$.
(iv) $z(2\tau) = z_0$.
(v) z *is even and* 2τ*-periodic.*

Proof Lemma 7(iii) implies that $r_1 < z_0 < r_2$. Hence (36) implies that, for all $t \in \mathbb{R}$, $z(t) \in (-\infty, r_1] \cup [z_0, r_2]$. Since z is continuous and $z(0) = z_0 \in [z_0, r_2]$, it follows that, for all $t \in \mathbb{R}$, $z(t) \in [z_0, r_2]$. It thus follows from similar arguments to those used in the proof of Proposition 9 that z has the required properties. □

Case 2.3: $\Delta = 0$. This case is characterized by $z_0 \neq z_e$ and $g(z_0) = z_e$.

In this case, Q has the multiple root r given by

$$r = \frac{z_0^2 + 1}{4z_e} = \sqrt{z_0^2 + 1} - z_0. \tag{68}$$

Lemma 11 *Assume that* $g(z_0) = z_e$. *Then, the following statements hold:*

(i) $r = z_0 \iff z_0 = 1/\sqrt{3} \iff z_0 = z_e$.
(ii) $r > z_0 \iff z_0 < 1/\sqrt{3}$.
(iii) $r < z_0 \iff z_0 > 1/\sqrt{3}$.
(iv) $z_0 \neq z_e \iff z_0 < z_e \iff \ddot{z}_0 < 0$.

Proof Statements (i)–(iii) are immediate, where the second implication in (i) follows from (34). To prove (iv), note that (34) shows that $z_0 \neq z_e$ if and only if $Q(z_0) \neq 0$, but, since Q is monic and has a repeated root, this is equivalent to $Q(z_0) > 0$ and, again, this is equivalent to $z_0 < z_e$. It follows from (i) that $z_0 \neq z_e$ if and only if $Q(z_0) \neq 0$, and thus $Q(z_0) > 0$. The second equivalence in (iv) follows from (22). □

So, assuming that $g(z_0) = z_e$, it follows from (29) that

$$\dot{z}(t)^2 = \frac{2a}{z(t)^2+1}(z(t)-r)^2(z_0 - z(t)), \tag{69}$$

which implies that, for all $t \in \mathbb{R}$, $z(t) \leq z_0$. We thus consider the following two subcases of Case 2.3.

Case 2.3.1: $\Delta = 0$ **and** $z_0 < \frac{1}{\sqrt{3}}$

Proposition 12 (Case 2.3.1) *Assume that* $g(z_0) = z_e$ *and* $z_0 < \frac{1}{\sqrt{3}}$. *Then,* $\dddot{z}_0 < 0$ *and, for all* $t \in \mathbb{R}$, $z(t) \leq z_0$. *Furthermore, as* $t \to \infty$, (37) *is satisfied and* $\lim_{t\to\infty} z(t) = -\infty$. *In addition, as* $t \to \infty$, (38) *is satisfied, where* Θ_∞ *is given by* (39).

Proof It follows from (ii) of Lemma 11 that $z_0 < r$. Now, (69) implies that $z(t) \leq z_0$, for all $t \in \mathbb{R}$. It follows that, $Q(z(t)) > 0$ for all $t \in \mathbb{R}$. The same discussion as in Case 2.1 shows that z decreases on $(0, \infty)$ and that, as $t \to \infty$, $z(t) \sim -\frac{a}{2}t^2$, and, as in Case 2.1, the angle of rotation of the wire converges to Θ_∞ given by (39). □

Case 2.3.2: $\Delta = 0$ **and** $z_0 > \frac{1}{\sqrt{3}}$

Proposition 13 (Case 2.3.2) *Assume that* $g(z_0) = z_e$ *and* $z_0 > \frac{1}{\sqrt{3}}$. *Then,* $\dddot{z}_0 < 0$, *z is decreasing on* $[0, +\infty)$ *and* $\lim_{t\to\infty} z(t) = r$, *where the rate of convergence is exponential.*

Proof Since $z_0 > \frac{1}{\sqrt{3}}$, Lemma 11 implies that $r < z_0 < z_e$, and $\dddot{z}_0 < 0$. Hence, there exists $\varepsilon > 0$ such that $\dot{z}$ is decreasing on $(0, \varepsilon)$. Now, define

$$t_1 \stackrel{\triangle}{=} \sup\{t > 0\colon z \text{ is decreasing on } (0, t)\}. \tag{70}$$

Note that $t_1 > \varepsilon$ and, for all $t \in (0, t_1)$, $\dot{z}(t) < 0$. It thus follows from (29) that, for all $t \in (0, t_1)$, $z(t) \neq r$, and thus, for all $t \in (0, t_1)$, $z(t) \in (-\infty, r) \cup (r, z_0)$. Since z is continuous and $z(0) = z_0$, it follows that, for all $t \in (0, t_1)$, $z(t) \in (r, z_0)$. Hence, for all $t \in (0, t_1)$,

$$\dot{z}(t) = (r - z(t))\sqrt{\frac{2a(z_0 - z(t))}{z(t)^2+1}}, \tag{71}$$

and thus, for all $t \in (0, t_1)$,

$$t = \int_{z(t)}^{z_0} \frac{1}{u-r}\sqrt{\frac{1+u^2}{2a(z_0-u)}}\,\mathrm{d}u. \tag{72}$$

Now, suppose that $t_1 < \infty$, which implies that $z(t_1) \geq r$. Note that the integral

$$\int_r^{z_0} \frac{1}{u-r}\sqrt{\frac{1+u^2}{2a(z_0-u)}}\,\mathrm{d}u \tag{73}$$

is divergent and thus cannot be equal to t_1. Hence, $z(t_1) > r$, and (71) implies that $\dot{z}(t_1) < 0$. It thus follows that there exists $\varepsilon > 0$ such that z decreases on $(t_1, t_1+\varepsilon)$, which contradicts the definition of t_1. Therefore, $t_1 = \infty$, and thus (72) implies that $\lim_{t\to\infty} z(t) = r$.

Next, to determine the rate of convergence of $z(t)$ as $t \to \infty$, note that (72) can be written as

$$\begin{aligned}
\sqrt{2a}t &= \int_{z(t)}^{z_0} \frac{1}{u-r}\sqrt{\frac{1+u^2}{z_0-u}}\,\mathrm{d}u \\
&= \int_{z(t)}^{z_0} \frac{1}{u-r}\left(\sqrt{\frac{1+u^2}{z_0-u}} - \sqrt{\frac{1+r^2}{z_0-r}}\right)\mathrm{d}u + \sqrt{\frac{1+r^2}{z_0-r}}\int_{z(t)}^{z_0}\frac{1}{u-r}\,\mathrm{d}u \\
&= \int_{z(t)}^{z_0} \frac{1}{u-r}\left(\sqrt{\frac{1+u^2}{z_0-u}} - \sqrt{\frac{1+r^2}{z_0-r}}\right)\mathrm{d}u - \sqrt{\frac{1+r^2}{z_0-r}}\log\left(\frac{z(t)-r}{z_0-r}\right).
\end{aligned}$$

Equivalently,

$$\sqrt{\frac{2a(z_0-r)}{1+r^2}}\,t = \int_{z(t)}^{z_0}\frac{g(u)-g(r)}{u-r}\,\mathrm{d}u - \log\left(\frac{z(t)-r}{z_0-r}\right),$$

where

$$g(u) = \sqrt{\frac{1+u^2}{1+r^2}\cdot\frac{z_0-r}{z_0-u}}. \tag{74}$$

Hence, defining

$$A(z_0) \triangleq \sqrt{\frac{2a(z_0-r)}{1+r^2}}, \quad B(z_0) \triangleq \int_r^{z_0}\frac{g(u)-g(r)}{u-r}\,\mathrm{d}u, \tag{75}$$

it follows that

$$\log\left(\frac{z(t)-r}{z_0-r}\right) + A(z_0)t - B(z_0) = -\int_r^{z(t)}\frac{g(u)-g(r)}{u-r}\,\mathrm{d}u. \tag{76}$$

Since $\lim_{t\to\infty} z(t) = r$, it follows that

$$\lim_{t\to\infty}\log\left(\frac{z(t)-r}{z_0-r}e^{A(z_0)t-B(z_0)}\right) = 0, \tag{77}$$

which implies that, as $t \to \infty$,

$$z(t) - r \sim (z_0 - r)e^{B(z_0)-A(z_0)t}. \tag{78}$$

Next, since

$$\lim_{u \to r^+} \frac{g(u) - g(r)}{u - r} = g'(r), \tag{79}$$

it follows that there exist $M > 0$ and $t_0 \geq 0$ such that, for all $t > t_0$,

$$\sup_{r<u<z(t)} \left| \frac{g(u) - g(r)}{u - r} \right| \leq M. \tag{80}$$

Hence, for all $t > t_0$, (78) implies that there exists $M' \geq M$ such that

$$\left| \int_r^{z(t)} \frac{g(u) - g(r)}{u - r} \, \mathrm{d}u \right| \leq M(z(t) - r) \leq M' e^{-A(z_0)t}.$$

Therefore, (76) implies that, as $t \to \infty$,

$$\log\left(\frac{z(t) - r}{z_0 - r}\right) + A(z_0)t - B(z_0) = \mathcal{O}(e^{-A(z_0)t}). \tag{81}$$

Finally, (81) implies that

$$z(t) - r = (z_0 - r)e^{B(z_0)-A(z_0)t}[1 + \mathcal{O}(e^{-A(z_0)t})],$$

or, equivalently,

$$z(t) = r + (z_0 - r)e^{B(z_0)-A(z_0)t} + \mathcal{O}(e^{-2A(z_0)t}).$$

□

5 Summary of Cases

Table 1 summarizes the seven cases considered in prior sections. In Fig. 2 the horizontal axis represents z_0, while the vertical axis represents z_e. The (z_0, z_e) plane is partitioned into seven regions, where each region is associated with one of the seven cases listed in Table 1.

Figure 2 characterizes the stability of the constant solution $z \equiv z_e = \frac{1}{\eta\omega_0^2}$. Fixing ω_0, (i.e., z_e), and letting z_0 vary, the horizontal line shown in Fig. 3 traverses the following cases:

(a) $z_e = \zeta < \frac{1}{\sqrt{3}}$. If $z_0 < \zeta$ but near ζ, then we are in Case 2.2.1, and, if $\zeta < z_0$, then we are in Case 2.2.3. See Fig. 3a.

Table 1 Assuming that $\dot{z}_0 = 0$, these seven cases classify the motion of the bead along the wire according to the initial conditions $(z_0, z_e) = (z_0, 1/\eta\omega_0^2)$, where the constant η depends only on the system given by (19), $g(z_0)$ is given by (32), and z_e is defined by (20)

Case	Result	Condition	Bead along wire
1	Proposition 2	$z_0 = z_e$	Is fixed
2.1	Proposition 5	$g(z_0) < z_e$	Diverges to $-\infty$
2.2.1	Proposition 8	$g(z_0) > z_e$ and $z_0 < \min(z_e, \frac{1}{\sqrt{3}})$	Diverges to $-\infty$
2.2.2	Proposition 9	$g(z_0) > z_e$ and $\frac{1}{\sqrt{3}} < z_0 < z_e$	Oscillates
2.2.3	Proposition 10	$g(z_0) > z_e$ and $z_e < z_0$	Oscillates
2.3.1	Proposition 12	$g(z_0) = z_e$ and $z_0 < 1/\sqrt{3}$	Diverges to $-\infty$
2.3.2	Proposition 13	$g(z_0) = z_e$ and $z_0 > 1/\sqrt{3}$	Converges

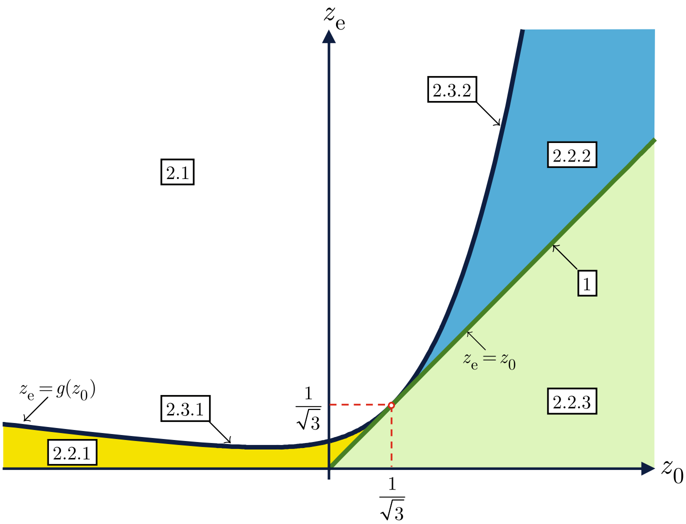

Fig. 2 This figure defines seven regions in the (z_0, z_e) plane. Each region corresponds to one of the cases listed in Table 1

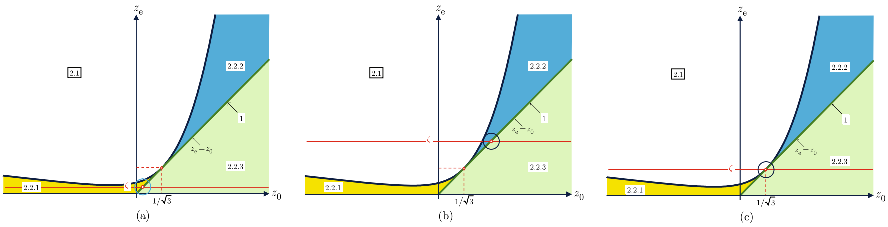

Fig. 3 **a** $z_e = \zeta < \frac{1}{\sqrt{3}}$. **b** $z_e = \zeta > \frac{1}{\sqrt{3}}$. **c** $z_e = \frac{1}{\sqrt{3}}$

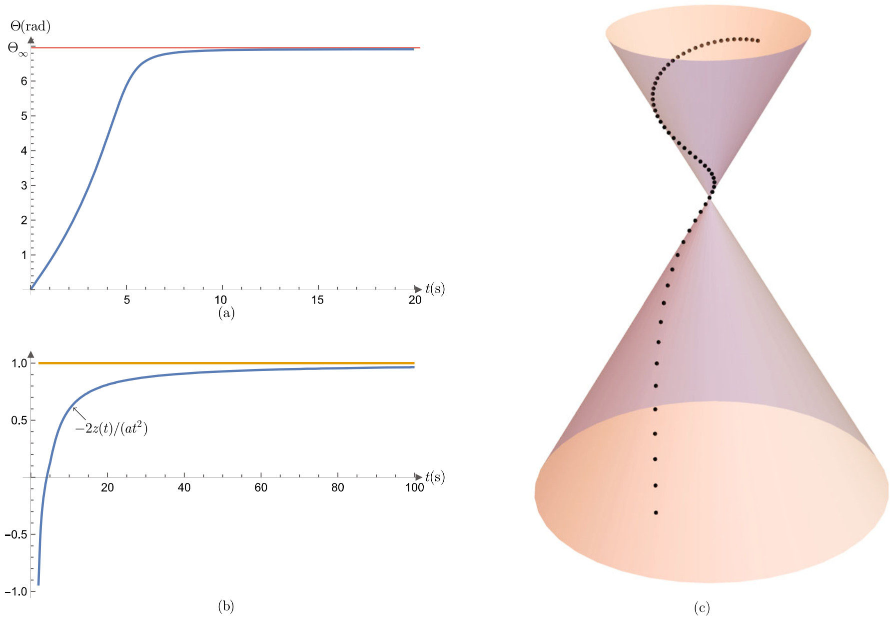

Fig. 4 Case 2.1: **a** $t \mapsto \Theta(t)$. **b** $t \mapsto \frac{z(t)}{(-at^2/2)}$. **c** The bead trajectory in space

(b) $z_e = \zeta > \frac{1}{\sqrt{3}}$. If $z_0 < \zeta$ but near ζ, then we are in Case 2.2.2, and, if $\zeta < z_0$, then we are in Case 2.2.3. See Fig. 3b.
(c) $z_e = \zeta = \frac{1}{\sqrt{3}}$. If $z_0 < \zeta$ but near ζ, then we are in Case 2.1, and, if $\zeta < z_0$, then we are in Case 2.2.3. See Fig. 3c.

6 Numerical Examples

This section presents numerical examples to illustrate the results obtained in previous sections. In all examples, we assume that $\eta = 1\,\text{s}^2$ and $\phi = \pi/3$ rad.

- **Case 2.1.** We consider $z_0 = 1$, $\omega_0 = 0.8\,\text{rad}\cdot\text{s}^{-1}$, so that $a = \frac{25}{64} = 0.390625$. Next, (23) is integrated over the interval $[0, 1000]$ using the NDSolve command from Wolfram Mathematica 13.3. This solution is then used to compute Θ over the same interval. Assuming that Θ_∞ is well approximated by $\Theta(1000)$, we obtain $\Theta_\infty \approx 6.91565$, while $\Theta(10) = 6.88653$ and $\Theta(100) = 6.91563$. Figure 4a shows Θ in terms of t in the interval $[0, 20]$. This shows that the solution is equivalent to $-\frac{a}{2}t^2$ for large t. In addition, Fig. 4b shows the ratio $-\frac{2}{at^2}z(t)$ over the interval $[2, 100]$. Finally, Fig. 4c provides a three-dimensional view of the motion of the bead as t varies in the interval $[0, 6]$, where the cone represents the surface of revolution swept by the wire.
- **Case 2.2.1** We consider $z_0 = 1$, $\omega_0 = 1.41\,\text{rad}\cdot\text{s}^{-1}$, so that $a = 0.32179$. Next, (23) is integrated over the interval $[0, 200]$ using NDSolve. We then use

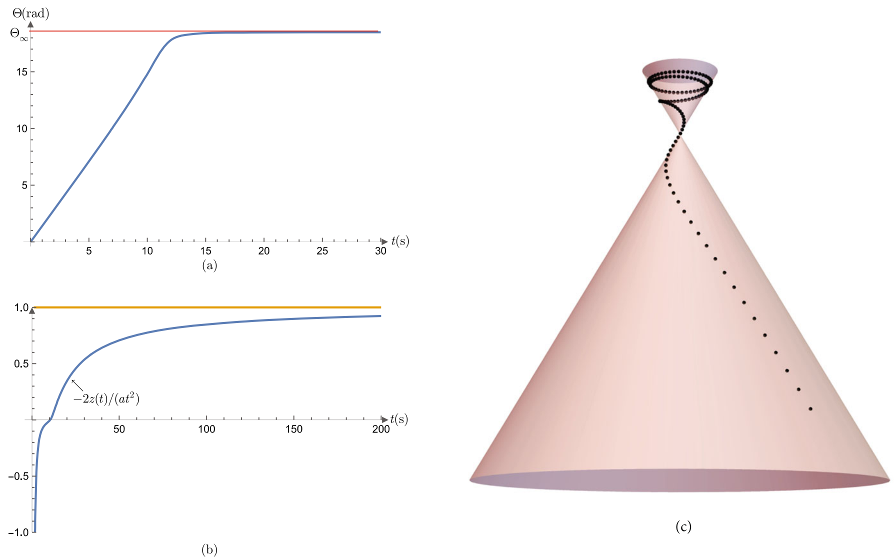

Fig. 5 Case 2.2.1: **a** $t \mapsto \Theta(t)$. **b** $t \mapsto \frac{z(t)}{(-at^2/2)}$. **c** The bead trajectory in space

this solution to compute Θ over the same interval. Assuming that Θ_∞ is well approximated by $\Theta(200)$, we obtain $\Theta_\infty \approx 8.49448$, while $\Theta(10) = 14.8189$ and $\Theta(100) = 18.49446$. Figure 5a shows Θ in terms of t in the interval $[0, 30]$. The computed trajectory shows that the solution is equivalent to $-\frac{a}{2}t^2$ for large t. Figure 5b shows the ratio $-\frac{2}{at^2}z(t)$ over the interval $[2, 200]$. Finally, Fig. 5c provides a three-dimensional view of the motion of the bead as t varies in the interval $[0, 13]$, where the cone represents the surface of revolution swept by the wire.

- **Case 2.2.2** We consider $z_0 = 2$, $\omega_0 = 0.5\,\mathrm{rad} \cdot \mathrm{s}^{-1}$, so that $a = 0.017778$. In this case we have

$$r_2 = \frac{5 + \sqrt{89}}{16} \approx 0.9021 \quad \text{and} \quad r_1 = \frac{5 - \sqrt{89}}{16} \approx -0.2771.$$

Hence, z is periodic with half-period τ given by (63). Numerical integration yields $\tau = 7.49916$. Numerical integration of (23) over the interval $[0, 5\tau]$ using NDSolve is shown in Fig. 6a, where the periodic nature of the solution z coincides with the theoretical value of the period. Finally, Fig. 6b presents a three-dimensional view of the motion of the bead as t varies in the interval $[0, 5\tau]$.

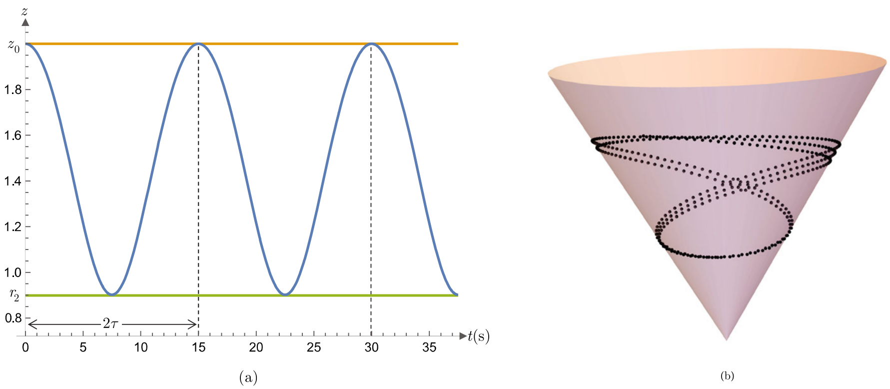

Fig. 6 Case 2.2.2: **a** $t \mapsto z(t)$. **b** The bead trajectory in space

7 Stability Analysis of the Equilibrium Solution

As an alternative approach to analyzing stability, it follows from (21) and (23) that

$$\ddot{z} = \frac{z}{(z^2+1)^2} - \frac{z_e}{(z_0^2+1)^2}. \tag{82}$$

Note that (82) is not a conventional ordinary differential equation since the right-hand side depends on the initial condition z_0. Nevertheless, z_e is an equilibrium solution of (82), and we can linearize (82) to determine the stability of this equilibrium. To do this, we define

$$z_1 \triangleq z, \tag{83}$$

$$z_2 \triangleq \dot{z}, \tag{84}$$

and rewrite (23) as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ \frac{z_1}{(z_1^2+1)^2} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{z_e}{(z_0^2+1)^2} \end{bmatrix}. \tag{85}$$

The following result is based on Lyapunov's first method.

Proposition 14 *The unique equilibrium solution of* (85) *is*

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_e \\ 0 \end{bmatrix}. \tag{86}$$

Furthermore, the associated linearized dynamics are given by $\dot{x} = Ax$, *where*

$$A \triangleq \begin{bmatrix} 0 & 1 \\ \frac{1-3z_e^2}{(z_e^2+1)^2} & 0 \end{bmatrix}. \tag{87}$$

Define

$$\lambda \triangleq \sqrt{\frac{|1-3z_e^2|}{(z_e^2+1)^2}}, \tag{88}$$

and let $\lambda_{1,2}$ *denote the eigenvalues of* A. *Then, the following statements hold:*

(i) *Assume that* $z_e < \frac{1}{\sqrt{3}}$. *Then,* $\lambda_1 = -\lambda < 0 < \lambda_2 = \lambda$, *and* (86) *is unstable.*

(ii) *Assume that* $z_e = \frac{1}{\sqrt{3}}$. *Then,* $\lambda_1 = \lambda_2 = 0$, *these eigenvalues are defective, and* (86) *is unstable.*

(iii) *Assume that* $z_e > \frac{1}{\sqrt{3}}$. *Then,* $\lambda_{1,2} \pm \jmath\lambda$.

Note that case (iii) is inconclusive (Khalil 2014). However, Fig. 2 shows that solutions corresponding to z_0 in a neighborhood of the ray $z_e = z_0 > 0$ are oscillatory and thus bounded. This suggests that case (iii) in Proposition 14 corresponds to Lyapunov stability. However, a complete analysis depends on consideration of $\dot{z}_0 \neq 0$.

8 Conclusions and Future Research

This paper analyzed the motion of a bead sliding along a frictionless, slanted, freely rotating wire. Under the assumption that the bead velocity along the wire is initially zero, it was shown that the bead trajectory may be constant, divergent, oscillatory, or convergent. Because angular momentum is conserved, the equation of motion of the bead depends on the initial bead position. By viewing the constant bead trajectory as an equilibrium solution, Lyapunov's direct method shows that, in two cases, the equilibrium is unstable, whereas the remaining case is inconclusive. Future research will focus on a more complete stability analysis of this system. Relevant techniques are discussed in Marsden and Scheurle (1993).

Appendix

In this section, we prove several inequalities used in the paper. Recall that, for all $u \in \mathbb{R}$,

$$g(u) = \frac{1+u^2}{4(\sqrt{1+u^2}-u)} = \tfrac{1}{4}(1+u^2)\sqrt{1+u^2}. \tag{89}$$

The following inequality is immediate.

A.0. *Let* $u \in \mathbb{R}$. *Then*

$$g(u) > 0. \tag{90}$$

A.1. *Let* $u > 0$. *Then*

$$u \leq g(u) \tag{91}$$

with equality if and only if $u = 1/\sqrt{3}$.

Proof For $u \neq 1/\sqrt{3}$, note that

$$u < g(u) \iff 4\sqrt{1+u^2} - 4u < u + \frac{1}{u}$$

$$
\begin{aligned}
&\iff 4\sqrt{1+u^2} < 5u + \frac{1}{u} \\
&\iff 16u^2 + 16 < 25u^2 + 10 + \frac{1}{u^2} \quad \text{(since both sides are positive)} \\
&\iff 0 < 9u^4 - 6u^2 + 1 \\
&\iff 0 < (3u^2 - 1)^2.
\end{aligned}
$$

Equality holds if and only if $u = 1/\sqrt{3}$. □

A.2. *Let* $u \in \mathbb{R}\backslash\{0\}$. *Then*

$$
g(u) \leq \frac{1+u^2}{4u} \iff 0 < u \leq \frac{1}{\sqrt{3}}, \tag{92}
$$

$$
g(u) \geq \frac{1+u^2}{4u} \iff (u < 0) \vee (u \geq \frac{1}{\sqrt{3}}). \tag{93}
$$

Proof To prove (92), note that

$$
\begin{aligned}
g(u) \leq \frac{1+u^2}{4u} &\iff \sqrt{1+u^2} + u \leq \frac{1}{u} \\
&\iff (u > 0) \text{ and } 1 + u^2 \leq \left(\frac{1}{u} - u\right)^2 \\
&\iff (u > 0) \text{ and } (3u^2 \leq 1) \\
&\iff 0 < u \leq \frac{1}{\sqrt{3}}.
\end{aligned}
$$

□

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Declarations

Conflict of interest The authors have no conflict of interest.

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