Differential Equations and Dynamical Systems (April 2024) 32(2):363–375
https://doi.org/10.1007/s12591-021-00574-8

ORIGINAL RESEARCH

The Dynamics of a Bead Sliding on a Freely Rotating Horizontal Wire: An Analytical Solution

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Accepted: 2 June 2021 / Published online: 2 July 2021

Abstract

This paper analyzes the dynamics of a bead sliding along a rotating horizontal wire whose rate of rotation varies due to the motion of the bead. Legendre's elliptic integrals are used to obtain an analytical solution of the equation of motion given by a nonlinear second-order differential equation. The analytical solution is used to determine the terminal velocity of the bead and the asymptotic angle of the wire.

Introduction

In applications of mechanics involving rotational motion, it is often convenient to formulate Newton's laws in a rotating frame. Within this noninertial setting, it is convenient to define centrifugal and Coriolis forces. Despite the fact that these "forces" play a palpable role in many engineering and scientific applications, they are not real forces, and thus are appropriately called fictitious [10, p. 317]. A substantial amount of discussion has been devoted to this topic [1]. The present paper considers a dynamics problem, namely, a bead sliding on a rotating wire, in which fictitious forces play a prominent role.

The motion of a bead sliding on a thin rigid wire has been studied in many contexts. The case of a circular wire rotating around a vertical diameter was considered in [2, p. 87] and further investigated in [7], and the case of a circular wire rotating around a horizontal diameter was studied in [8]. Using fractional calculus, the case of a straight wire inclined by a constant angle and rotating around a vertical axis was studied in [4]. The case of a heavy bead sliding on a parabolic wire rotating around its vertical axis was studied in [5, 9, 12], where approximations of the solutions are obtained using fractional calculus as well as homotopy perturbation and Laplace transforms. In all of these cases, the rate of rotation of the wire is assumed to be constant, and thus each of these mechanical systems has one degree of freedom.

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The present paper analyzes a mechanical system consisting of a bead sliding along a straight wire that freely rotates around a fixed point. In this system, the centrifugal force, which is aligned with the wire, plays a visually compelling role; however, the real force is orthogonal to the wire. Since the rate of rotation of the wire varies due to the motion of the bead, the considered system has two degrees of freedom, namely, the angle through which the wire has rotated and the position of the bead along the wire. However, because of conservation of energy and conservation of angular momentum, the dynamics of the system are given by a second-order ODE. The main contribution of this paper is an exact analytical solution of the differential equations governing the motion of the system in terms of elliptic functions. The analytical solution is used to prove that the wire asymptotically comes to rest and provides an explicit expression for the asymptotic angle of the wire as a function of the initial angular velocity.

The contents of the paper are as follows. In Sect. 2 we formulate the problem and fix some notation. In Sect. 3 the nonlinear ODE is analyzed, and qualitative properties of its solutions are derived. Section 4 derives an analytical expression for the solution of the ODE in terms of Legendre's elliptic integrals. Finally, in Sect. 5 we draw some conclusions and propose some further research.

Problem Formulation

Consider an infinitely long straight wire that rotates frictionlessly in a horizontal plane about a fixed point. A bead slides without friction along the wire, moving away from the fixed point as the wire rotates (see Fig. 1). Since the plane of rotation is horizontal, gravity has no effect on either the bead or the wire. The angular rate of rotation of the wire is not prescribed, but rather is determined by its interaction with the bead.

The motion of the bead along the wire suggests that a force is acting along the wire. The only force on the bead, however, is the reaction force between the bead and the wire, but this force is *orthogonal* to the wire rather than parallel to it. The apparent contradiction between the direction of the force and the direction of the acceleration is explained by Newton's second law which states that the *inertial* acceleration of a particle is proportional to the applied force. The inertial acceleration of the bead is thus orthogonal to the wire, whereas the acceleration along the wire is *noninertial*. Noninertial acceleration is proportional to a fictitious force, which, in this case, is the centrifugal force.

A convenient technique for obtaining the equations of motion for a mechanical system without using Newton's second law is given by Lagrangian dynamics. In particular, the kinetic energy of the bead-and-wire system is given by

$$T(\dot{\theta}, x, \dot{x}) = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{\theta}^2 x^2), \tag{1}$$

where, as shown in Fig. 1, θ is the rotation angle of the wire relative to a reference direction, x is the distance from the fixed point of the wire to the bead, $J > 0$ is the moment of inertia of the wire about its fixed point, and $m > 0$ is the mass of the bead. Lagrangian dynamics states that the equations of motion for the bead-and-wire system are given by [3, p. 20]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = 0, \tag{2}$$

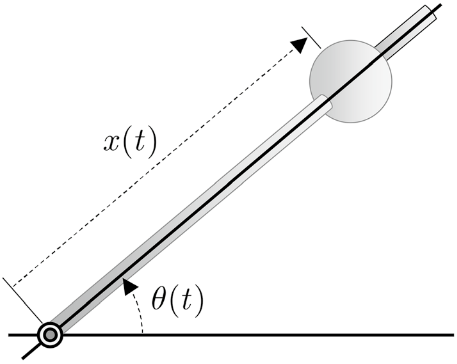

Fig. 1 The bead sliding on the infinite wire viewed from above

where $q_1 = \theta$ and $q_2 = x$. Since $\partial T/\partial\theta = 0$, it follows from (2) that θ satisfies

$$\dot{\theta}(t) = \frac{c}{J + mx(t)^2}, \tag{3}$$

where c is a constant, and that x satisfies

$$\ddot{x}(t) = \dot{\theta}(t)^2 x(t). \tag{4}$$

For simplicity, we consider the initial conditions $x(0) > 0$, $\dot{x}(0) > 0$, and $\dot{\theta}(0) > 0$, which imply that $c = (J + mx(0)^2)\dot{\theta}(0) > 0$. It turns out that c is the angular momentum of the system. Hence, defining $L(t) \stackrel{\triangle}{=} (J + mx(t)^2)\dot{\theta}(t)$, it follows from (3) that, for all $t \geq 0$,

$$L(t) = L(0). \tag{5}$$

This is conservation of angular momentum, which reflects the absence of external moments.

Equations (3) and (4) can be merged into the single nonlinear second-order differential equation

$$\ddot{x}(t) = \frac{c^2 x(t)}{(J + mx(t)^2)^2}, \tag{6}$$

which is the focus of this article. Some properties of the solution $x(t)$ of (6) can be inferred, at least heuristically, by observation. For example, since the initial position $x(0)$ is positive, it follows from continuity that there exists $t_1 > 0$ such that, for all $t \in [0, t_1]$, $x(t)$ is positive. Integrating (6), it follows that, for all $t \in [0, t_1]$, the velocity of the bead satisfies

$$\dot{x}(t) = \int_0^t \frac{c^2 x(s)}{(J + mx(s)^2)^2}\,\mathrm{d}s > 0, \tag{7}$$

which implies that the position of the bead is increasing on $[0, t_1]$. Furthermore, it follows from (6) that the acceleration of the bead is also positive, and thus the velocity of the bead is increasing on $[0, t_1]$. Since the bead speeds up on the interval $[0, t_1]$, repeating this argument starting at t_1 suggests that the direction of motion of the bead never reverses and that its position diverges to infinity; this will subsequently be proven to be the case. In addition, (3) implies that the angular velocity $\dot{\theta}(t)$ of the wire is decreasing, and thus, since the position of the bead diverges, $\dot{\theta}(t)$ converges to zero. More can be said. Since energy is conserved and the energy $E(t)$ of the system at time t is the kinetic energy $T(\dot{\theta}(t), x(t), \dot{x}(t))$, it follows that, during the motion,

$$E(t) = E(0), \tag{8}$$

where, using (3),

$$\begin{aligned} E(t) &= \tfrac{1}{2} J\dot{\theta}(t)^2 + \tfrac{1}{2} m(\dot{x}(t)^2 + \dot{\theta}^2(t)x^2(t)) \\ &= \tfrac{1}{2} c\dot{\theta}(t) + \tfrac{1}{2} m\dot{x}(t)^2. \end{aligned} \tag{9}$$

Solving for $\dot{x}(t)$ yields

$$\dot{x}(t) = \sqrt{\frac{2E(0) - c\dot{\theta}(t)}{m}}, \tag{10}$$

and thus, since $\dot{\theta}(t)$ converges to 0, it follows that $\dot{x}(t)$ converges to the terminal velocity v_∞ given by

$$v_\infty \stackrel{\triangle}{=} \sqrt{\frac{2E(0)}{m}}. \tag{11}$$

One of the goals of this article is to investigate the asymptotic behavior of $x(t)$ and $\theta(t)$. We are especially interested in the asymptotic motion of the wire, which cannot be inferred from the above observations. In particular, the property $\dot{\theta}(t) \to 0$ as $t \to \infty$ does not imply that $\lim_{t\to\infty} \theta(t)$ exists, that is, the angle of the wire may or may not converge. In fact, although the function $f(x) = \sqrt{x}$ has the property that $f'(x) \to 0$ as $x \to \infty$, it is also true that $f(x) \to \infty$ as $x \to \infty$.

Note that, since angular momentum is conserved and is nonzero, the wire cannot stop rotating since otherwise $L(t)$ would be zero. Rather, conservation of angular momentum implies that, as the angular velocity of the wire converges to zero, the distance from the bead to the fixed point is increasing in such a way that $L(t)$ remains constant. We can thus expect a delicate relationship between the position of the bead—whose speed is increasing—and the angle of the wire—whose angular velocity is decreasing.

Analysis of the Nonlinear ODE

To simplify (6), define $y(t) \stackrel{\triangle}{=} \lambda x(\mu t)$, where λ and μ are positive constants specified below. It thus follows that

$$\begin{aligned} \ddot{y}(t) &= \lambda\mu^2 \ddot{x}(\mu t) \\ &= \lambda\mu^2 \left(\frac{c}{J + mx(\mu t)^2} \right)^2 x(\mu t) \\ &= \frac{\mu^2 c^2 \lambda^4 / m^2}{(J\lambda^2/m + y(t)^2)^2} y(t). \end{aligned}$$

Defining

$$\lambda \overset{\triangle}{=} \sqrt{\frac{m}{J}}, \qquad \mu \overset{\triangle}{=} \frac{J}{c}, \tag{12}$$

it follows that y satisfies the differential equation

$$\ddot{y} = \frac{y}{(1+y^2)^2}, \tag{13}$$

with the initial conditions $y(0) = \lambda x(0) > 0$ and $\dot{y}(0) = \lambda\mu\dot{x}(0) > 0$. The following result concerns the existence and uniqueness of solutions to (13).

Proposition 1 *Let* $(t_0, y_0, y_0') \in \mathbb{R}^3$. *Then there exists a unique solution* y *of* (13) *defined on* $[t_0, \infty)$ *and that satisfies the initial conditions* $y(t_0) = y_0$ *and* $y'(t_0) = y_0'$.

Proof Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(u, v) \overset{\triangle}{=} (v, u/(1+u^2)^2)$. Setting $Y = (y, y')$, (13) takes the form of the Cauchy problem $Y' = F(Y)$, which is concerned with solutions of (13) satisfying the initial condition $Y(t_0) = (y_0, y_0')$.

Since F is continuously differentiable, it is locally Lipschitz, and thus [6, Theorem 2.6] implies the existence of a unique solution of (13) in a neighborhood of t_0.

Next, let $\Phi : [t_0, T) \to \mathbb{R}^2$ be the maximal solution of $Y' = F(Y)$ that satisfies the initial conditions $\Phi(t_0) = (y_0, y_0')$, where $T \in (t_0, \infty) \cup \{\infty\}$. We now show that $T = \infty$.

First, note that

$$\langle (u, v), F(u, v) \rangle = uv\left(1 + \frac{1}{(1+u^2)^2}\right) \le 2|uv| \le u^2 + v^2 = \|(u, v)\|^2, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^2$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Defining $h(t) \overset{\triangle}{=} \|\Phi(t)\|^2$, (14) implies

$$h'(t) = 2\langle \Phi(t), \Phi'(t) \rangle = 2\langle \Phi(t), F(\Phi(t)) \rangle \le 2\|\Phi(t)\|^2 = 2h(t).$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(h(t)e^{-2t}) = (h'(t) - 2h(t))e^{-2t}$, it follows that $t \mapsto \|\Phi(t)\|^2 e^{-2t}$ is nonincreasing on $[t_0, T)$ and thus, for all $t \in [t_0, T), \|\Phi(t)\| \le \left\|\Phi(t_0)\right\| e^{t-t_0}$. Now, suppose that T is finite. Then $t \mapsto \|\Phi(t)\|$ is bounded by $\left\|\Phi(t_0)\right\| e^{T-t_0}$ on $[t_0, T)$, and the continuity of F on $\mathbb{R}^2$ implies that $t \mapsto \Phi'(t) = F(\Phi(t))$ is also bounded on $[t_0, T)$. It thus follows that there exists $M \in (0, \infty)$ such that, for all $s, t \in \mathbb{R}$ satisfying $t_0 \le s \le t < T$,

$$\|\Phi(t) - \Phi(s)\| = \left\| \int_s^t \Phi'(u)\,\mathrm{d}u \right\| \le \int_s^t \left\|\Phi'(u)\right\| \mathrm{d}u \le M(t-s). \tag{15}$$

The Cauchy criterion for the existence of a limit of a function implies that $\widetilde{\Phi} \overset{\triangle}{=} \lim_{t\to T} \Phi(t)$ exists. In addition, the solution Ψ of $Y' = F(Y)$ that satisfies $\Psi(T) = \widetilde{\Phi}$ yields an extension of Φ to an interval $[t_0, T')$, where $T' > T$. This contradicts the maximality of Φ. Hence $T = \infty$. □

Proposition 1 can be proved more directly by invoking known results. For example, since F is locally Lipschitz and bounded, Theorem 4.6 of [11] implies that every solution exists on $[t_0, \infty)$. Alternatively, since F is globally Lipschitz, Theorem 4.8 of [11] implies that every solution exists on $[t_0, \infty)$.

Proposition 1 leads to the following definition, which will be useful for subsequent analysis.

Definition 2 For all $(u, v) \in (0, \infty)^2$, let $F_{u,v}$ denote the unique function $F_{u,v} : [0, \infty) \to \mathbb{R}$ satisfying

$$\ddot{F}_{u,v}(t) = \frac{F_{u,v}(t)}{(1 + F_{u,v}(t)^2)^2}, \qquad F_{u,v}(0) = u, \quad \dot{F}_{u,v}(0) = v. \tag{16}$$

Proposition 3 *Let $(u, v) \in (0, \infty)^2$. Then $F_{u,v}$ is positive, strictly convex, and increasing on $[0, \infty)$. In addition, for all $t \geq 0$, $F_{u,v}(t) \geq u + vt$, and thus $\lim_{t\to\infty} F_{u,v}(t) = \infty$.*

Proof Writing F for $F_{u,v}$ and defining $f(t) \stackrel{\triangle}{=} F(t)\dot{F}(t)$, it follows that

$$\dot{f}(t) = \dot{F}(t)^2 + F(t)\ddot{F}(t) = \dot{F}(t)^2 + \frac{F(t)^2}{(1 + F(t)^2)^2} \geq 0.$$

Hence, f is nondecreasing on $[0, \infty)$. Since $f(0) = uv > 0$, it follows that f is positive on $[0, \infty)$. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}t}F^2(t) = 2f(t)$ implies that F^2 is increasing on $[0, \infty)$ with $F(0)^2 = u^2 > 0$. Hence, for all $t \in [0, \infty)$, $F(t)$ is nonzero, and, since $F(0) = u > 0$, F is positive on $[0, \infty)$. Consequently, $\ddot{F} = F/(1 + F^2)^2$ is also positive on $[0, \infty)$, and thus F is strictly convex on $[0, \infty)$. In addition, since f and F are positive on $[0, \infty)$, it follows that $\dot{F} = f/F$ is positive on $[0, \infty)$, and thus F is increasing on $[0, \infty)$. Next, since F is strictly convex, the graph of F lies above the line tangent to the graph at $t = 0$. Therefore, for all $t \geq 0$, $F(t) \geq F(0) + \dot{F}(0)\,t = u + vt$. Finally, $v > 0$ implies that $\lim_{t\to\infty} F(t) = \infty$. □

Proposition 4 *Let $(u, v) \in (0, \infty)^2$, and define*

$$w \triangleq \sqrt{v^2 + \frac{1}{1 + u^2}}. \tag{17}$$

Then, for all $t \in [0, \infty)$,

$$\dot{F}_{u,v}(t)^2 + \frac{1}{1 + F_{u,v}(t)^2} = w^2. \tag{18}$$

In particular, for all $t \geq 0$, $\dot{F}_{u,v}(t) < w$, and

$$\lim_{t\to\infty} \dot{F}_{u,v}(t) = w. \tag{19}$$

Proof Writing F for $F_{u,v}$ and denoting the left hand side of (18) by $h(t)$, it follows that

$$\dot{h}(t) = 2\left(\ddot{F}(t) - \frac{F(t)}{(1 + F(t)^2)^2}\right)\dot{F}(t) = 0.$$

Therefore, for all $t \geq 0$, $h(t) = h(0) = w^2$. Furthermore, since, for all $t \geq 0$, $\dot{F}(t) > 0$, it follows that $\dot{F}(t) < \sqrt{h(t)} = w$. Finally, since Proposition 3 implies that $F(t) \geq u + vt$, $F(t)$ diverges to infinity as $t \to \infty$, which, using (18), implies (19). □

Applying l'Hôpital's rule to (19) yields

$$\lim_{t\to\infty} \frac{F_{u,v}(t)}{t} = w.$$

However, the following stronger result holds.

Proposition 5 *For all* $(u, v) \in (0, \infty)^2$, *the integral*

$$\kappa(u, v) \triangleq \int_0^\infty \frac{\mathrm{d}s}{(w + \dot{F}_{u,v}(s))(1 + F_{u,v}(s)^2)} \tag{20}$$

is convergent. Furthermore, the function $t \mapsto F_{u,v}(t) - wt$ *is decreasing on* $[0, \infty)$ *from* u *to* $u - \kappa(u, v)$ *and converges to* $u - \kappa(u, v)$ *as* $t \to \infty$. *In particular, for all* $t \geq 0$,

$$wt + u - \kappa(u, v) \leq F_{u,v}(t) \leq wt + u. \tag{21}$$

Proof Writing F for $F_{u,v}$, Proposition 3 implies that $\ddot{F}$ is positive on $[0, \infty)$, and thus $\dot{F}$ is increasing on $[0, \infty)$. It thus follows from Proposition 4 that, for all $s \geq 0$, $v = \dot{F}(0) \leq \dot{F}(s) < w$, and thus

$$\frac{1}{(w + \dot{F}(s))\dot{F}(s)} \leq \frac{1}{v(w + v)}.$$

Therefore, for all $s \geq 0$,

$$\frac{1}{(w + \dot{F}(s))(1 + F(s)^2)} \leq \frac{1}{v(w + v)} \cdot \frac{\dot{F}(s)}{1 + F(s)^2}. \tag{22}$$

Integrating (22) implies that, for all $t \geq 0$,

$$\int_0^t \frac{\mathrm{d}s}{(w + \dot{F}(s))(1 + F(s)^2)} \leq \frac{1}{v(w + v)}(\arctan(F(t)) - \arctan(u)) < \frac{\pi}{2v(v + w)},$$

which proves that (20) converges.

Next, using (18), it follows that, for all $t \geq 0$,

$$\begin{aligned} wt + u - F(t) &= \int_0^t (w - \dot{F}(s))\, \mathrm{d}s \\ &= \int_0^t \frac{w^2 - \dot{F}(s)^2}{w + \dot{F}(s)}\, \mathrm{d}s \\ &= \int_0^t \frac{\mathrm{d}s}{(w + \dot{F}(s))(1 + F(s)^2)}. \end{aligned}$$

Hence, for all $t \geq 0$,

$$F(t) = wt + u - \int_0^t \frac{\mathrm{d}s}{(w + \dot{F}(s))(1 + F(s)^2)}. \tag{23}$$

The convergence of the integral in (23) and the positivity of the integrand imply that $t \mapsto F(t) - wt$ decreases on $[0, \infty)$ from u to $u - \kappa(u, v)$. □

Using Elliptic Integrals to Obtain an Analytical Solution

In this section, we obtain an analytical expression for the solution $x(t)$ of (6). This solution is given in terms of Legendre's elliptic integrals; properties of these functions are given in [13, Chapter 19]. Asymptotic analysis of this analytical solution recovers the terminal velocity (11) and, by using (3), provides an expression for the limiting angle, thus proving that the wire comes to rest. This result strengthens the analysis in Section 1, which showed only that $\lim_{t\to\infty} \dot{\theta}(t) = 0$.

For $\mu > 0$ and $0 \le \phi < \arcsin(1/\max\{1, \mu\})$, Legendre's elliptic integrals of the first and second kind $\mathcal{F}(\phi, \mu)$ and $\mathcal{E}(\phi, \mu)$, respectively, are defined by

$$\mathcal{F}(\phi, \mu) = \int_0^\phi \frac{1}{\sqrt{1-\mu^2\sin^2\eta}}\,\mathrm{d}\eta, \tag{24}$$

$$\mathcal{E}(\phi, \mu) = \int_0^\phi \sqrt{1-\mu^2\sin^2\eta}\,\mathrm{d}\eta. \tag{25}$$

For convenience, we also define

$$\mathcal{D}(\phi, \mu) = \int_0^\phi \frac{\sin^2\eta}{\sqrt{1-\mu^2\sin^2\eta}}\,\mathrm{d}\eta, \tag{26}$$

which is related to $\mathcal{E}$ and $\mathcal{F}$ by

$$\mu^2\mathcal{D}(\phi, \mu) = \mathcal{F}(\phi, \mu) - \mathcal{E}(\phi, \mu). \tag{27}$$

The following lemma will be needed.

Lemma 6 *For* $\mu > 0$ *and* $0 \le \psi < \phi < \arcsin(1/\max\{1, \mu\})$,

$$\int_\psi^\phi \frac{\mathrm{d}\eta}{\sin^2\eta\sqrt{1-\mu^2\sin^2\eta}} = H(\phi, \mu) - H(\psi, \mu), \tag{28}$$

where

$$H(\eta, \mu) \stackrel{\triangle}{=} -\cot\eta\sqrt{1-\mu^2\sin^2\eta} + \mu^2\mathcal{D}(\eta, \mu). \tag{29}$$

Proof For all $\eta \in [0, \arcsin(1/\max\{1, \mu\}))$,

$$\begin{aligned}
\frac{\partial H}{\partial \eta}(\eta, \mu) &= \frac{1}{\sin^2\eta}\sqrt{1-\mu^2\sin^2\eta} + \cot\eta\frac{\mu^2\sin\eta\cos\eta}{\sqrt{1-\mu^2\sin^2\eta}} + \frac{1}{\sqrt{1-\mu^2\sin^2\eta}} - \sqrt{1-\mu^2\sin^2\eta} \\
&= \cot^2\eta\sqrt{1-\mu^2\sin^2\eta} + \frac{\mu^2\cos^2\eta}{\sqrt{1-\mu^2\sin^2\eta}} + \frac{1}{\sqrt{1-\mu^2\sin^2\eta}} \\
&= \frac{\cot^2\eta - \mu^2\cos^2\eta + \mu^2\cos^2\eta + 1}{\sqrt{1-\mu^2\sin^2\eta}} \\
&= \frac{1}{\sin^2\eta\sqrt{1-\mu^2\sin^2\eta}},
\end{aligned}$$

which implies (28). □

Since $F_{u,v}$ defines an increasing C^1 diffeomorphism from $[0, \infty)$ to $[u, \infty)$, the inverse function theorem can be used to define $G_{u,v} \triangleq F_{u,v}^{-1} : [u, \infty) \to [0, \infty)$. In fact, Proposition 4 and Lemma 6 provide an expression for the inverse function $G_{u,v}$ in terms of the elliptic integrals.

Proposition 7 *Let $(u, v) \in (0, \infty)^2$ and $z \geq u$. Then*

$$G_{u,v}(z) = -\frac{1}{w} H\left(\arctan\left(\frac{1}{z}\right), \frac{1}{w}\right) + \frac{1}{w} H\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right), \tag{30}$$

where H is defined by (29) and w is defined by (17). Furthermore, as $z \to \infty$,

$$G_{u,v}(z) = \frac{z}{w} - \frac{uv}{w^2} + \frac{1}{w^3} \mathcal{D}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) + O\left(\frac{1}{z}\right). \tag{31}$$

Proof Proposition 4 implies that

$$\dot{F} = \sqrt{w^2 - \frac{1}{1 + F^2}} = \sqrt{\frac{w^2 - 1 + w^2 F^2}{1 + F^2}}$$

and thus

$$\sqrt{\frac{1 + F^2}{w^2 - 1 + w^2 F^2}} \dot{F} = 1$$

Equivalently,

$$\int_{F(0)}^{F(t)} \sqrt{\frac{1 + s^2}{w^2 - 1 + w^2 s^2}} \, \mathrm{d}s = t.$$

Finally, using $t = G_{u,v}(z)$ yields

$$\int_{u}^{z} \sqrt{\frac{1 + s^2}{w^2 - 1 + w^2 s^2}} \, \mathrm{d}s = G_{u,v}(z). \tag{32}$$

The change of variables $s = \cot \eta$ yields

$$\begin{aligned} G_{u,v}(z) &= \frac{1}{w} \int_{\arctan(1/z)}^{\arctan(1/u)} \frac{\mathrm{d}\eta}{\sin^2 \eta \sqrt{1 - w^{-2} \sin^2 \eta}} \\ &= \frac{H(\arctan(1/u), 1/w) - H(\arctan(1/z), 1/w)}{w}, \end{aligned}$$

and (30) follows. Next, note that

$$\begin{aligned} H\left(\arctan\left(\frac{1}{z}\right), \frac{1}{w}\right) &= -z \cdot \sqrt{1 - \frac{1}{w^2(1 + z^2)}} + \frac{1}{w^2} \mathcal{D}\left(\arctan\left(\frac{1}{z}\right), \frac{1}{w}\right), \\ H\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) &= -\frac{uv}{w} + \frac{1}{w^2} \mathcal{D}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right), \end{aligned}$$

and since, as $z \to \infty$, $\mathcal{D}(\arctan(1/z), 1/w) = O(1/z)$, it follows that

$$z + H\left(\arctan\left(\frac{1}{z}\right), \frac{1}{w}\right) = z\left(1 - \sqrt{1 - \frac{1}{w^2(1+z^2)}}\right) + O\left(\frac{1}{z}\right) = O\left(\frac{1}{z}\right).$$

Therefore, as $z \to \infty$,

$$G_{u,v}(z) = \frac{z}{w} - \frac{uv}{w^2} + \frac{1}{w^3}\mathcal{D}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) + O\left(\frac{1}{z}\right),$$

as required. □

Corollary 8 *Let* $(u, v) \in (0, \infty)^2$. *Then, as* $t \to \infty$,

$$F_{u,v}(t) = wt + \frac{uv}{w} - \mathcal{F}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) + \mathcal{E}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) + O\left(\frac{1}{t}\right), \tag{33}$$

where w *is defined by* (17).

Proof Replacing $z = F_{u,v}(t)$ in the asymptotic expansion (31) of $G_{u,v}$ and recalling that, as $t \to \infty$, $F_{u,v}(t) \sim wt$, it follows that

$$t = \frac{F_{u,v}(t)}{w} - \frac{uv}{w^2} + \frac{1}{w^3}\mathcal{D}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right) + O\left(\frac{1}{t}\right),$$

which is equivalent to (33). □

Corollary 9 *Let* $(u, v) \in (0, \infty)^2$. *Then*

$$\int_0^\infty \frac{\mathrm{d}t}{1 + F_{u,v}(t)^2} = \frac{1}{w}\mathcal{F}\left(\arctan\left(\frac{1}{u}\right), \frac{1}{w}\right). \tag{34}$$

Proof Since $F_{u,v}(t) \sim wt$ as $t \to \infty$, the integral (34) is convergent. Denoting this integral by $J(u, v)$, the change of variables $z = F_{u,v}(t)$ yields

$$J(u, v) = \int_u^\infty \frac{G'_{u,v}(z)}{1 + z^2}\,\mathrm{d}z. \tag{35}$$

Furthermore, (32) implies that

$$G'_{u,v}(z) = \sqrt{\frac{1 + z^2}{w^2(1+z^2) - 1}}. \tag{36}$$

Hence,

$$J(u, v) = \int_u^\infty \sqrt{\frac{1 + z^2}{w^2(1+z^2) - 1}} \cdot \frac{\mathrm{d}z}{1 + z^2}. \tag{37}$$

Now, the change of variables $z = \cot\eta$ shows that

$$J(u,v) = \frac{1}{w}\int_0^{\arctan(1/u)} \frac{1}{\sqrt{1-w^{-2}\sin^2\eta}}\mathrm{d}\eta$$
$$= \frac{1}{w}\mathcal{F}\Big(\arctan\Big(\frac{1}{u}\Big), \frac{1}{w}\Big),$$

as required. □

Returning to the original Eqs. (3) and (4), note that the solution x of (4) is related to F by

$$x(t) = \frac{1}{\lambda}F_{\lambda x(0),\lambda\mu\dot{x}(0)}\left(\frac{t}{\mu}\right) = \sqrt{\frac{J}{m}}F_{\sqrt{m/J}x(0),\sqrt{mJ}\dot{x}(0)/c}\left(\frac{ct}{J}\right),$$

with $c = L(0)$, which yields the following result.

Corollary 10 *Let* $(x_0, \dot{x}_0, \dot{\theta}_0) \in (0,\infty)^3$, *and let* θ *and* x *denote the solutions of* (3) *and* (4) *with the initial conditions* $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, *and* $\dot{\theta}(0) = \dot{\theta}_0$. *Then, as* $t \to \infty$,

$$x(t) = v_\infty t + B + O\Big(\frac{1}{t}\Big),$$

with

$$B \triangleq \frac{x_0\dot{x}_0}{v_\infty} - \sqrt{\frac{J}{m}}\Big(\mathcal{F}(\arctan(\alpha),\phi) - \mathcal{E}(\arctan(\alpha),\phi)\Big), \tag{38}$$

and the dimensionless constants $\alpha \triangleq \sqrt{\frac{J}{mx_0^2}}$ *and* $\phi \triangleq \frac{L(0)}{\sqrt{2JE(0)}}$. *Furthermore,*

$$\lim_{t\to\infty}\dot{x}(t) = v_\infty, \tag{39}$$

$$\lim_{t\to\infty}\dot{\theta}(t) = 0, \tag{40}$$

$$\lim_{t\to\infty}\theta(t) = \theta_\infty, \tag{41}$$

with

$$\theta_\infty = \theta(0) + \phi\,\mathcal{F}(\arctan(\alpha),\phi). \tag{42}$$

This final result follows from Corollary 9 and

$$\lim_{t\to\infty}\theta(t) = \theta(0) + \int_0^\infty \frac{c}{J + mx(t)^2}\,\mathrm{d}t$$
$$= \theta(0) + \frac{c}{J}\int_0^\infty \frac{1}{1 + F_{u,v}(ct/J)^2}\,\mathrm{d}t$$
$$= \theta(0) + \int_0^\infty \frac{1}{1 + F_{u,v}(s)^2}\,\mathrm{d}s,$$

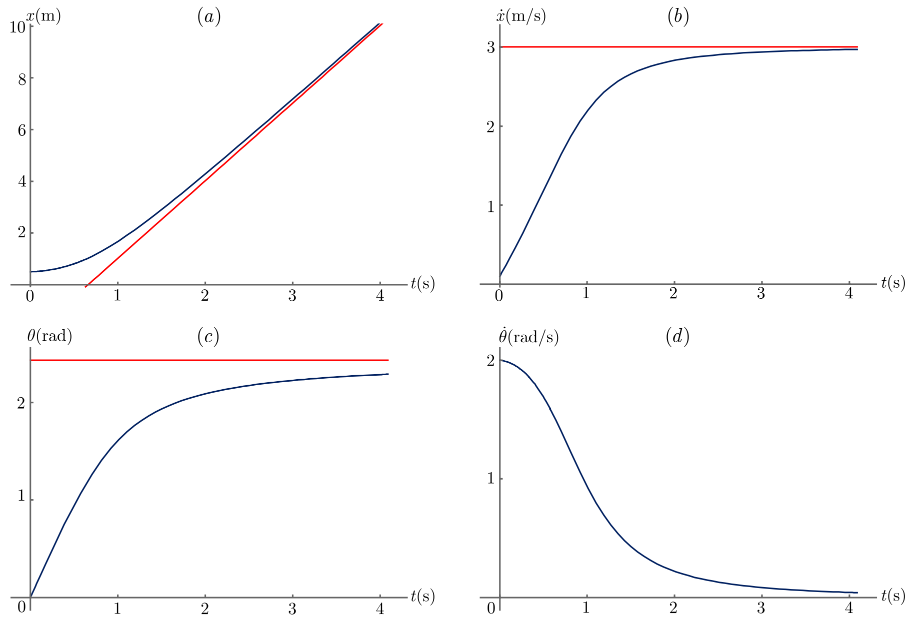

Fig. 2 Numerical example with the parameter values $m = 1\,\text{kg}$, $J = 2\,\text{kg}\cdot\text{m}^2$ and the initial conditions $x(0) = 0.5\,\text{m}$, $\dot{x}(0) = 0.1\,\text{m/s}$, $\theta(0) = 0\,\text{rad}$, and $\dot{\theta}(0) = 2\,\text{rad/s}$. **a** Shows the position $x(t)$ of the bead and its asymptote $v_\infty t + B$, **b** shows the velocity $\dot{x}(t)$ of the bead and its terminal velocity v_∞, **c** shows the angle $\theta(t)$ of the wire relative to its initial orientation and the limiting angle, and **d** shows the angular velocity $\dot{\theta}(t)$ of the wire

where $u = \sqrt{m/J}x_0$ and $v = \sqrt{mJ}\dot{x}_0/c$. Indeed, Corollary 9 shows that the wire does not rotate endlessly, but rather approaches an angle determined by the initial conditions.

Finally, viewing the instant t as the starting point, applying (42) yields

$$\theta_\infty = \theta(t) + \phi\,\mathcal{F}\left(\arctan\left(\sqrt{\frac{J}{m(x(t))^2}}\right), \phi\right). \tag{43}$$

Hence, $\theta(t)$ is given as a function of $x(t)$ by

$$\theta(t) = \theta(0) + \phi\left(\mathcal{F}(\arctan(\alpha), \phi) - \mathcal{F}\left(\arctan\left(\sqrt{\frac{J}{m(x(t))^2}}\right), \phi\right)\right). \tag{44}$$

Figure 2 illustrates $x(t), \dot{x}(t), \theta(t)$, and $\dot{\theta}(t)$ as functions of t for a given set of parameters.

Conclusions and Extensions

The analytical solution derived in this paper can be applied to the case where $x(0)$, $\dot{x}(0)$, and $\dot{\theta}(0)$ have arbitrary signs, and thus the bead may reverse direction. However, the asymptotic properties of the solution are unchanged.

Another extension is to introduce gravity and consider the case where the wire is bent at the fixed point so that a component of gravity acts along the wire.

Finally, for circular and parabolic wires, it is of interest to consider the case where the wire rotates freely. Although all of the cases entail two degrees of freedom, conservation of energy and angular momentum result in a single second-order differential equation for the position of the bead.

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