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Self-Excited Dynamics of Discrete-Time Lur'e Systems With Affinely Constrained, Piecewise-C^1 Feedback Nonlinearities

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ABSTRACT Self-excited systems (SES) arise in numerous applications, such as fluid-structure interaction, combustion, and biochemical systems. In support of system identification and digital control of SES, this paper analyzes discrete-time Lur'e systems with affinely constrained, piecewise-C^1 feedback nonlinearities. In particular, a novel feature of the discrete-time Lur'e system considered in this paper is the structural assumption that the linear dynamics possess a zero at 1. This assumption ensures that the Lur'e system have a unique equilibrium for each constant, exogenous input and prevents the system from having an additional equilibrium with a nontrivial domain of attraction. The main result provides sufficient conditions under which a discrete-time Lur'e system is self-excited in the sense that its response is 1) nonconvergent for almost all initial conditions, and 2) bounded for all initial conditions. Sufficient conditions for 1) include the instability and nonsingularity of the linearized, closed-loop dynamics at the unique equilibrium and their nonsingularity almost everywhere. Sufficient conditions for 2) include asymptotic stability of the linear dynamics of the Lur'e system and their feedback interconnection with linear mappings that correspond to the affine constraints that bound the nonlinearity, as well as the feasibility of a linear matrix inequality.

INDEX TERMS Discrete-time, Lur'e system, nonlinear feedback, self-excitation, self-oscillation.

I. INTRODUCTION

Self-excited systems (SES) have the property that constant inputs lead to oscillatory outputs [1]. The diversity of applications in which SES arise is vast, and encompasses fluid-structure interaction [2], thermoacoustic oscillations [3], and chemical and biochemical systems [4]. Not surprisingly, extensive effort has been devoted to modeling and controlling SES [5], [6]. SES are also used for controller tuning; for PID control, a relay inserted inside a servo loop induces limit-cycle oscillations, which are used to identify the crossover frequency [7].

Control of SES requires analytical and empirical models; the present paper is motivated by the latter need. System identification for SES based on continuous-time Lur'e models is considered in [8]. Alternatively, for sampled-data control, system identification for SES based on discrete-time Lur'e models is considered in [9]. In support of discrete-time system identification and sampled-data control of SES, the present paper focuses on discrete-time Lur'e models of SES.

A Lur'e system consists of linear dynamics with memoryless nonlinear feedback. The stability of Lur'e systems is a classical problem, expressed by the Aizerman conjecture for sector-bounded nonlinearities [10], [11], [12]. Although the Aizerman conjecture is false, the stability of Lur'e systems has been widely studied in both continuous time [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] and discrete time [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46].

In contrast to stable behavior, many SES are modeled by Lur'e systems that have unstable equilibria and bounded response. A classical example is the Rijke tube, in which

TABLE 1. Lur'e system literature.

	Asymptotically Stable	Bounded and Nonconvergent
CT	[13]–[17]	[18]–[27]
DT	[28]–[42]	[43]–[46]

acoustic waves interact through feedback with the flame dynamics to produce thermoacoustic oscillations [3]. Self-excited oscillations in continuous-time Lur'e systems have been studied in [19], [20], [21], [23], [24], [26], [27]. In particular, using the bounded real lemma, continuous-time Lur'e systems with superlinear feedback and minimum-phase linear dynamics with relative degree 1 or 2 are shown in [24] to possess bounded solutions. Related results are given in [26] based on dissipativity theory as well as in [27] using Lyapunov methods.

In contrast to [19], [20], [21], [23], [24], [26], [27], the present paper focuses on discrete-time, self-excited Lur'e systems, with the property that, for all constant inputs, the response is 1) nonconvergent for almost all initial conditions, and 2) bounded for all initial conditions. In this work, 1) is equivalent to stating that the set of initial conditions for which the state trajectory is convergent has measure zero; intuitively, this implies that the probability of randomly selecting an initial state for which the state trajectory is convergent is zero. The main contribution of the present paper is sufficient conditions for this behavior for a specific class of nonlinear feedback functions. The analogous property for continuous-time Lur'e systems is not addressed in the literature.

It is important to stress the distinctions between continuous-time and discrete-time Lur'e systems that exhibit self-excited behavior. In particular, since superlinear feedback has unbounded gain, the linear dynamics of a continuous-time Lur'e system must be high-gain stable. From a root locus perspective, this means that the linear dynamics must be minimum phase, the relative degree cannot exceed 2, and, when the relative degree is 2, the root locus center must lie in the open left half plane. These assumptions, which are invoked in [24] for continuous-time dynamics, do not imply high-gain stability for discrete-time systems with strictly proper linear dynamics. As discussed in [35], bounded response of a discrete-time Lur'e system with superlinear feedback requires positive-real, and thus relative-degree-zero, linear dynamics. Superlinear feedback is thus incompatible with discrete-time Lur'e systems.

Table 1 categorizes some of the literature on continuous-time (CT) and discrete-time (DT) Lur'e systems in terms of asymptotically stable response and bounded, nonconvergent response. The most relevant among these works to the present paper are [43], [44], [45], [46] on discrete-time Lur'e systems that have bounded, nonconvergent response. In particular,

- Reference [43] extends the frequency-domain results of [19] to discrete-time Lur'e systems with a sector-bounded nonlinearity and provides sufficient conditions under which the Lur'e system yields a self-excited response. Note that it follows from (5) of [43] that bounded nonlinearities, such as saturation functions, do not meet the assumptions for the existence of a self-excited response.
- Reference [44] uses a graphical tool based on Hopf bifurcation analysis, the harmonic balance method, and the Nyquist stability criterion to determine self-excited responses arising from discrete-time Lur'e systems with a smooth nonlinearity.
- Reference [45] considers a discrete-time Lur'e system with a washout filter, a time delay, and a saturation nonlinearity, and uses frequency-domain results to provide sufficient conditions for the existence of a nonconstant periodic response.
- Reference [46] considers a discrete-time Lur'e system and uses frequency-domain results to provide sufficient conditions for the existence of a slope-restricted nonlinearity under which a nonconstant periodic response with a specific frequency exists. As stated in Remark 2 of [46], the robustness of the periodic response to initial-condition variations is not guaranteed. Hence, under these assumptions, the set of initial conditions that do not give rise to the periodic response may not have measure zero.

None of these works, however, provide sufficient conditions under which a discrete-time Lur'e system is self-excited in the sense of the present paper, as discussed later in this section.

In order to address the special features of self-excited discrete-time Lur'e systems, the main contribution of the present paper is to prove that a class of discrete-time Lur'e systems with *affinely constrained* feedback are self-excited in the sense that 1) the set of initial conditions for which the state trajectory is convergent has measure zero, and 2) all trajectories are bounded. Although an affinely constrained function need not be bounded or even sector-bounded, it must have linear growth, thus ruling out superlinear nonlinearities, as necessitated by the fact that discrete-time strictly proper linear systems are not high-gain stable. By bounding the slope of the linear growth of the feedback nonlinearity, the linear-growth assumption enables self-oscillating discrete-time Lur'e systems with unbounded feedback nonlinearities. As a benefit of this setting, the linear discrete-time dynamics of the Lur'e system need not be minimum phase, which is assumed in [24] for continuous-time systems. Note that the affine constraint of the feedback nonlinearities is applied only to values corresponding to the domain outside of a neighborhood of the origin.

An additional novel feature of the discrete-time Lur'e system considered in this paper is the structural assumption that the linear dynamics possess a zero at 1. This assumption, which places a washout filter in the loop, blocks the DC component arising from the constant exogenous input to the system and ensures that the nonlinear closed-loop system has a unique equilibrium for each constant, exogenous input. Most importantly, this property prevents the Lur'e system from having an additional equilibrium with a nontrivial domain of attraction. Furthermore, by blocking the DC component of the signal, the washout filter tends to center the output around zero, thus taking advantage of the destabilizing property of the derivative of the nonlinearity at zero, as expressed by the spectral radius assumption in the statement of Theorem 3.6.

One of the main contributions of the present paper is Theorem 3.6, which provides sufficient conditions under which the set of initial conditions for which the trajectories of the Lur'e system are convergent has measure zero. Theorem 3.6 is applicable to discrete-time Lur'e systems with piecewise-C^1 nonlinearities for which the derivative of the closed-loop dynamics may be singular on a set of measure zero. The need to consider piecewise-C^1 nonlinearities is motivated by their role in nonlinear system identification [9], [47], [48], [49]. Note that in the case where feedback nonlinearity is C^1 and the derivative of the closed-loop dynamics is everywhere nonsingular, Theorem 3.6 follows from Theorem 2 in [50], which studies the divergence of equilibria rather than self-excited behavior.

The main result of the present paper is Theorem 3.9, which depends on Theorem 3.6 and provides sufficient conditions for the Lur'e system to be self-excited. Unlike [43], [44], [45], [46], Theorem 3.9 does not use frequency-domain results. Furthermore, note that affinely constrained, piecewise-C^1 nonlinearities, as defined in Definition 3.7, include bounded, non-smooth, and saturation nonlinearities, which implies that Theorem 3.9 provides results for systems not considered in [43], [44] and extends the results of [45]. Finally, unlike [46], Theorem 3.9 makes no assumption about the frequency content of the self-excited response and provides sufficient conditions under which the set of initial conditions that do not give rise to the periodic response have measure zero.

The contents of the paper are as follows. Section II introduces the discrete-time Lur'e system, which involves asymptotically stable linear dynamics in feedback with a memoryless nonlinearity, and analyzes its equilibrium properties. Section III defines affinely constrained nonlinearities and provides sufficient conditions under which the discrete-time Lur'e system possesses a bounded, nonconvergent response for almost all initial conditions. Finally, Section IV presents a numerical example that illustrates the assumptions for self-excitation presented in Section III. Fig. 1 shows the dependencies of the results in this paper.

Nomenclature and terminology: $\mathbb{R} \triangleq (-\infty, \infty)$, $\mathbb{N} \triangleq \{1, 2, \ldots\}$, $\mathbb{C}$ denotes the complex numbers, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$, and $\mathbf{z} \in \mathbb{C}$ denotes the Z-transform variable. For $\mathcal{G} \subseteq \mathbb{R}^n$, $\mathrm{cl}(\mathcal{G})$ denotes the closure of $\mathcal{G}$, $\dim(\mathcal{G})$ denotes the dimension of $\mathcal{G}$, and, as stated in Definition 3.1, $\mathrm{iso}(\mathcal{G})$ denotes the set of isolated

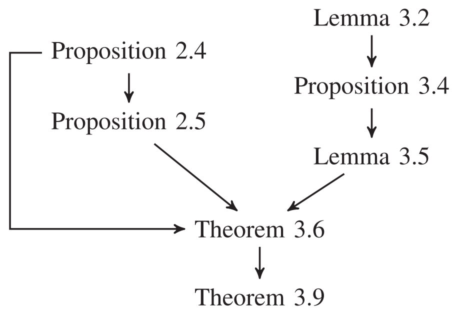

FIGURE 1. **Result dependencies.**

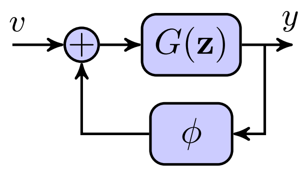

FIGURE 2. **Discrete-time Lur'e system.**

points of $\mathcal{G}$, and $\mathrm{acc}(\mathcal{G})$ denotes the set of accumulation points of $\mathcal{G}$. For $\mathcal{G} \subseteq \mathbb{R}^n$, $\mathcal{H} \subseteq \mathbb{R}^m$, and $F \in \mathbb{R}^{m\times n}$, define the mapping $F\mathcal{G} \triangleq \{Fx \in \mathbb{R}^m : x \in \mathcal{G}\}$, the inverse mapping $F^{-1}\mathcal{H} \triangleq \{x \in \mathbb{R}^n : Fx \in \mathcal{H}\}$, and the kernel of a linear map $\ker F \triangleq \{x \in \mathbb{R}^n : Fx = 0\}$. For $\mathcal{G} \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n, f^1 \triangleq f, f^{-1}(\mathcal{G}) \triangleq \{x \in \mathbb{R}^n : f(x) \in \mathcal{G}\}$, and, for all $k \geq 1, f^{k+1} \triangleq f \circ f^k, f^{-k-1}(\mathcal{G}) \triangleq f^{-1}(f^{-k}(\mathcal{G}))$. For (Lebesgue) measurable $\mathcal{G} \subseteq \mathbb{R}^n$, $\mu(\mathcal{G})$ denotes the measure of $\mathcal{G}$. For $x \in \mathbb{R}^n$ and $\varepsilon > 0, \mathbb{B}_\varepsilon(x)$ denotes the open ball of radius ε centered at x. Positive-definite matrices are assumed to be symmetric. For $A \in \mathbb{R}^{n\times n}$, $\mathrm{spr}(A)$ denotes the spectral radius of $A, \|A\|$ denotes the maximum singular value of A, and, if $A$is positive definite, then $\lambda_{\min}(A)$ denotes the eigenvalue of A of minimum magnitude and $\lambda_{\max}(A)$ denotes the eigenvalue of A of maximum magnitude. The terminology "$\lim_{k\to\infty} \alpha_k$ exists" implies that the indicated limit is finite. The (Fréchet) derivative of $f : \mathbb{R}^m \to \mathbb{R}^n$ is the $n \times m$ matrix (that is, the Jacobian) denoted by $f'(x)$.

II. ANALYSIS OF THE LUR'E SYSTEM

Let $G(\mathbf{z}) = C(\mathbf{z}I - A)^{-1}B$ be a nonzero, strictly proper, discrete-time, single-input single-output (SISO) transfer function, and let (A, B, C) be a realization of G with state $x_k \in \mathbb{R}^n$ at step k, let $\phi : \mathbb{R} \to \mathbb{R}$, and let $v \in \mathbb{R}$. The results of this paper do not require that (A, B, C) be a minimal realization of G; however, since G is not the zero transfer function, it follows that B is nonzero and thus injective and C is nonzero and thus surjective. Then, for all $k \geq 0$, the discrete-time Lur'e system in Fig. 2 has the closed-loop dynamics

$$x_{k+1} = Ax_k + B(\phi(y_k) + v), \tag{1}$$

$$y_k = Cx_k, \tag{2}$$

and thus

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} B(\phi(y_i) + v), \tag{3}$$

$$y_k = Cx_k = CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-1-i} B(\phi(y_i) + v). \tag{4}$$

Note that (1), (2) can be written as

$$x_{k+1} = f(x_k), \tag{5}$$

where $f(x) \stackrel{\triangle}{=} Ax + B(\phi(Cx) + v)$. Note that, in the case where ϕ is continuous, f is continuous and thus measurable, which in turn implies that, for all $k \geq 1$, f^k is measurable.

Definition 2.1: (1), (2) is *self-excited* if, for all $v \in \mathbb{R}$, the following hold:

i) For almost all $x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} x_k$ does not exist.

ii) For all $x_0 \in \mathbb{R}^n$, $(x_k)_{k=1}^{\infty}$ is bounded.

Note that *i*) and *ii*) are the discrete-time analog of "oscillations in the sense of Yakubovich" [21], [27], [51], [52]. Furthermore, note that *i*) holds if and only if $\mu(\{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k \text{ exists}\}) = 0$. The following result concerns the measure of the set of initial conditions for which the output converges.

Proposition 2.2: Assume that $\mathrm{spr}(A) < 1$ and ϕ is continuous, and let $v \in \mathbb{R}$. If $\mu(\{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k \text{ exists}\}) = 0$, then $\mu(\{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} y_k \text{ exists}\}) = 0$.

Proof: Let $v \in \mathbb{R}$, and define

$$X_0 \stackrel{\triangle}{=} \{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k \text{ exists}\},$$

$$Y_0 \stackrel{\triangle}{=} \{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} y_k \text{ exists}\}.$$

Note that X_0 can be written as

$$X_0 = \bigcap_{\kappa\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n_1,n_2>N} \left\{x \in \mathbb{R}^n : \|f^{n_1}(x) - f^{n_2}(x)\| < \frac{1}{\kappa}\right\}. \tag{6}$$

Since f^k is measurable for all $k \geq 1$, and since the right-hand side of (6) constitutes countable unions and intersections of measurable sets, it follows that X_0 is measurable. For all $x_0 \in Y_0$, $\lim_{k\to\infty}(\phi(y_k)+v)$ exists, and thus, since $\mathrm{spr}(A) < 1$, it follows from (3) that, for all $x_0 \in Y_0$, $\lim_{k\to\infty} x_k$ exists, and thus $Y_0 \subseteq X_0$. Hence, in the case where $\mu(X_0) = 0$, $0 \leq \mu(Y_0) \leq \mu(X_0) = 0$, and thus $\mu(Y_0) = 0$. □

Definition 2.3: $x \in \mathbb{R}^n$ is an *equilibrium* of (1), (2) if x is a fixed point of f, that is,

$$x = f(x) = Ax + B(\phi(Cx) + v). \tag{7}$$

When $I - A$ is nonsingular, define

$$x_{\mathrm{e}} \stackrel{\triangle}{=} (I - A)^{-1} B v \tag{8}$$

and note that

$$Cx_{\mathrm{e}} = G(1)v. \tag{9}$$

The following result establishes useful properties of G, ϕ and the equilibria of (1), (2). These properties motivate assumptions that are invoked in the paper. Note that similar equilibria results arise when studying the converging-input converging-state property for discrete-time Lur'e systems, as show in [53].

Proposition 2.4: Assume that $I - A$ is nonsingular. Then, the following hold:

i) $x \in \mathbb{R}^n$ is an equilibrium of (1), (2) if and only if

$$x = (I - A)^{-1} B(\phi(Cx) + v). \tag{10}$$

ii) If $x \in \mathbb{R}^n$ is an equilibrium of (1), (2), then the following hold:

a) $Cx = G(1)(\phi(Cx) + v)$.

b) $\phi(Cx) = -v$ if and only if $x = 0$.

c) If $G(1) = 0$, then $Cx = 0$ and $x = (I - A)^{-1}B(\phi(0) + v)$ is the unique equilibrium of (1), (2).

d) If $Cx = 0$, then either $G(1) = 0$ or $v = -\phi(0)$.

e) If $\phi(Cx) = 0$, then $x = x_{\mathrm{e}}$.

iii) The following are equivalent:

a) x_{e} is an equilibrium of (1), (2).

b) $\phi(Cx_{\mathrm{e}}) = 0$.

c) $\phi(G(1)v) = 0$.

iv) If $G(1) \neq 0$, then the following are equivalent:

a) x_{e} is an equilibrium of (1), (2).

b) $\phi(Cx_{\mathrm{e}}) = 0$.

c) $v \in \frac{1}{G(1)}\phi^{-1}(\{0\})$.

v) If $G(1) = 0$, then the following are equivalent:

a) $\phi(0) = 0$.

b) x_{e} is an equilibrium of (1), (2).

c) x_{e} is the unique equilibrium of (1), (2).

The proof of Proposition 2.4 is given in the appendix. Note that the converse of Proposition 2.4-*ii)e)* is true and is given by *iii*).

In the following result, *i*) implies that every convergent state trajectory of (1), (2) converges to an equilibrium solution. Under stronger sufficient conditions, *ii*) implies that every convergent state trajectory of (1), (2) converges to the unique equilibrium solution given by (8).

Proposition 2.5: Assume that ϕ is continuous. Then, the following hold:

i) If $x_\infty \stackrel{\triangle}{=} \lim_{k\to\infty} x_k$ exists, then x_∞ is an equilibrium of (1), (2).

ii) If $I - A$ is nonsingular, $G(1) = 0$, and $\phi(0) = 0$, then the following hold:

a) If $x_\infty \stackrel{\triangle}{=} \lim_{k\to\infty} x_k$ exists, then $x_\infty = x_{\mathrm{e}}$.

b) $\{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k \text{ exists}\} = \{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k = x_{\mathrm{e}}\}$.

Proof: To prove *i*), note that, since ϕ is continuous, it follows that f is continuous, and thus $f(x_\infty)$ exists and $\lim_{x\to x_\infty} f(x) = f(x_\infty)$. Hence, (5) implies that $x_\infty = \lim_{k\to\infty} x_k = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k) = \lim_{x\to x_\infty} f(x) = f(x_\infty)$, and thus it follows from (7) that x_∞ is an equilibrium of (1), (2).

To prove *ii*)*a*), note that *i*) implies that x_∞ is an equilibrium of (1), (2). Since $I - A$ is nonsingular, $G(1) = 0$ and $\phi(0) = 0$, Proposition 2.4*v* implies that x_e is the unique equilibrium of (1), (2). Hence, $x_\infty = x_e$.

To prove *ii*)*b*), note that "$\subseteq$" follows from *ii*)*a*). Finally, "$\supseteq$" is immediate. □

III. SELF-EXCITED DYNAMICS OF THE LUR'E SYSTEM

This section presents sufficient conditions under which the Lur'e system (1), (2) with an affinely constrained nonlinearity is self-excited. Section III-A shows some preliminary results about accumulation points and the application of the inverse mapping of the linear operator C to a set with no accumulation points. Section III-B defines piecewise continuously differentiable functions, and presents results on the repeated application of the inverse mapping of f, with piecewise continuously differentiable ϕ, to a set of measure zero. Section III-C shows results about the nonconvergence of the solutions of the considered Lur'e system, which depend on the results introduced in Section III-B. Section III-D introduces the definition of affinely constrained functions. Finally, Section III-E presents results on the boundedness of the solutions of the Lur'e system, and uses these results along with results in Section III-C to show that the Lur'e system is self-excited under the given assumptions.

A. PRELIMINARY RESULTS

Definition 3.1: Let $\mathcal{B} \subseteq \mathbb{R}^n$. Then, $z \in \mathcal{B}$ is an *isolated point of* $\mathcal{B}$ if there exists $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(z) \cap (\mathcal{B}\backslash\{z\}) = \varnothing$. Furthermore, $z \in \mathbb{R}^n$ is an *accumulation point* of $\mathcal{B}$ if, for all $\varepsilon > 0, \mathbb{B}_\varepsilon(z) \cap (\mathcal{B}\backslash\{z\}) \neq \varnothing$. The set of accumulation points of $\mathcal{B}$ is denoted by $\mathrm{acc}(\mathcal{B})$, and the set of isolated points of $\mathcal{B}$ is denoted by $\mathrm{iso}(\mathcal{B})$.

Note that $z \in \mathrm{acc}(\mathcal{B})$ if and only if there exists $(x_i)_{i=1}^\infty \subseteq \mathcal{B}\backslash\{z\}$ such that $\lim_{i\to\infty} x_i = z$. Furthermore, note that $z \in \mathrm{acc}(\mathcal{B})$ need not be an element of $\mathcal{B}$. In fact, $\mathrm{cl}(\mathcal{B})\backslash\mathcal{B} \subseteq \mathrm{cl}(\mathcal{B})\backslash\mathrm{iso}(\mathcal{B}) = \mathrm{acc}(\mathcal{B})$, and thus $\mathrm{acc}(\mathcal{B}) = \varnothing$ if and only if $\mathcal{B} = \mathrm{iso}(\mathcal{B})$.

The following lemma applies the inverse mapping of C to a set with no accumulation points. The proof proceeds by first showing that the set with no accumulation points is countable, which is then used to show that applying the inverse mapping of C yields a countable set of hyperplanes of measure zero.

Lemma 3.2: Let $\mathcal{A} \subseteq \mathbb{R}$, assume that $\mathrm{acc}(\mathcal{A}) = \varnothing$, and define $\mathcal{B} \triangleq C^{-1}\mathcal{A} \subseteq \mathbb{R}^n$, and thus $\mathcal{A} = C\mathcal{B}$. Then, the following hold:

i) $\mu(\mathcal{B}) = 0$.
ii) $\mathcal{B}$ is closed.
iii) $\mathcal{B}$ is the countable union of translates of a proper subspace of $\mathbb{R}^n$.

Proof: *i*)–*iii*) are true when $\mathcal{A}$ is empty; hence assume $\mathcal{A}$ is not empty. First, we show that $\mathcal{A}$ is countable. Let $m \in \mathbb{N}$, and suppose that $\mathcal{A} \cap \mathbb{B}_m(0)$ is infinite. Since the closure of $\mathcal{A} \cap \mathbb{B}_m(0)$ is compact, it follows that $\mathcal{A} \cap \mathbb{B}_m(0)$ contains a convergent sequence whose limit is an accumulation point of $\mathcal{A}$, which is a contradiction. Hence, $\mathcal{A} \cap \mathbb{B}_m(0)$ has a finite number of elements. Thus, $\mathcal{A} = \cup_{k\in\mathbb{N}}(\mathcal{A} \cap \mathbb{B}_k(0))$ is the union of a countable number of finite sets and thus is countable.

To prove *i*), note that $\mathcal{B}$ can be written as

$$\mathcal{B} = \bigcup_{y\in\mathcal{A}} \mathcal{C}_y, \tag{11}$$

where, for all $y \in \mathcal{A}$, $\mathcal{C}_y \triangleq \{x \in \mathbb{R}^n : Cx = y\}$ is a hyperplane and $\mu(\mathcal{C}_y) = 0$. Since $\mathcal{A}$ is countable, it follows from (11) that

$$\mu(\mathcal{B}) = \mu\left(\bigcup_{y\in\mathcal{A}} \mathcal{C}_y\right) = \sum_{y\in\mathcal{A}} \mu(\mathcal{C}_y) = 0. \tag{12}$$

To prove *ii*), note that, since $\mathrm{acc}(\mathcal{A}) = \varnothing$, it follows that $\mathcal{A} = \mathrm{iso}(\mathcal{A})$, and thus $\mathcal{A}$ is closed. Hence, $\mathcal{B}$ is closed. Finally, to prove *iii*), since $\mathcal{A}$ is countable, write $\mathcal{A} = \{t_m \in \mathbb{R} : m \in \mathbb{N}\}$ and, for all $m \in \mathbb{N}$, let $x_m \in \mathbb{R}^n$ satisfy $Cx_m = t_m$. Then, $\mathcal{B} = \ker C + \{x_m \in \mathbb{R}^n : m \in \mathbb{N}\} = \cup_{m=1}^\infty (x_m + \ker C)$. Since it is assumed in Section II that $n \geq 2$, it follows from the rank-nullity theorem that $\dim(\ker C) = \dim(\mathbb{R}^n) - \dim(\mathbb{R}) = n - 1 \in (0, n)$. Hence, $\dim(\ker C) < n$ and $\dim(\ker C) \neq \{0\}$, and thus $\ker C$ is a proper subspace of $\mathbb{R}^n$. Therefore, $\mathcal{B}$ is the countable union of translates of a proper subspace of $\mathbb{R}^n$. □

B. PIECEWISE-C¹ FUNCTIONS

Definition 3.3: ϕ is *piecewise continuously differentiable* (PWC¹) if the following hold:

i) ϕ is continuous.
ii) Define $\mathcal{R} \triangleq \{y \in \mathbb{R} : \phi'(y)$ exists and ϕ' is continuous at $y\}$. Then, $\mathcal{S} \triangleq \mathbb{R}\backslash\mathcal{R}$ has no accumulation points.
iii) For all $y \in \mathcal{S}$, $\lim_{t\uparrow 0} \phi'(y + t)$ and $\lim_{t\downarrow 0} \phi'(y + t)$ exist.

Note that, if ϕ is C¹, then $\mathcal{S} = \varnothing$.

As an example, consider $\phi(y) = y^2 \sin(1/y)$ for $y \neq 0$ and $\phi(0) = 0$. Then, $\phi'(y) = 2y \sin(1/y) - \cos(1/y)$ for $y \neq 0$ and $\phi'(0) = 0$. Hence, $\mathcal{R} = \mathbb{R}\backslash\{0\}$ and $\mathcal{S} = \{0\}$. However, neither $\lim_{t\uparrow 0} \phi'(t)$ nor $\lim_{t\downarrow 0} \phi'(t)$ exist, and thus ϕ is not PWC¹.

Note that, in the case where $\phi'(y)$, $\lim_{t\uparrow 0} \phi'(y + t)$, and $\lim_{t\downarrow 0} \phi'(y + t)$ exist, it follows that $\phi'(y) = \lim_{t\uparrow 0} \phi'(y + t) = \lim_{t\downarrow 0} \phi'(y + t)$, and thus ϕ' is continuous at y. Therefore, if ϕ is PWC¹ and $y \in \mathcal{S}$, then $\phi'(y)$ does not exist, even though *iii*) holds. Furthermore, *ii*) holds if and only if each bounded subset of $\mathbb{R}$ contains a finite number of elements of $\mathcal{S}$.

Assume that ϕ is PWC¹. Then, since C is surjective, as mentioned in Section II, define $\mathcal{D} \triangleq C^{-1}\mathcal{R}$ and $\mathcal{E} \triangleq C^{-1}\mathcal{S} = \mathbb{R}^n\backslash\mathcal{D}$ so that $\mathcal{R} = C\mathcal{D}$ and $\mathcal{S} = C\mathcal{E}$. If $x \in \mathcal{D}$, then $f'(x) = A + \phi'(Cx)BC$. Note that, in the case where $G(1) = 0$, it follows that $f'(x_e) = f'(0) = A + \phi'(0)BC$. Finally, define

$$\mathcal{R}_0 \triangleq \{y \in \mathcal{R} : A + \phi'(y)BC \text{ is singular}\} \subseteq \mathcal{R}, \tag{13}$$

$$\mathcal{D}_0 \triangleq C^{-1}\mathcal{R}_0. \tag{14}$$

It thus follows that

$$\mathcal{D}_0 = \{x \in \mathcal{D} : f'(x) \text{ is singular}\}. \tag{15}$$

Proposition 3.4: Assume that ϕ is PWC^1 and acc$(\mathcal{R}_0) = \varnothing$. Then, $\mathcal{D}_0$ and $\mathcal{E}$ are closed and $\mu(\mathcal{D}_0) = \mu(\mathcal{E}) = 0$.

Proof: Since acc$(\mathcal{R}_0) =$ acc$(\mathcal{S}) = \varnothing, \mathcal{D}_0 = C^{-1}\mathcal{R}_0$, and $\mathcal{E} = C^{-1}\mathcal{S}$, *i*) and *ii*) of Lemma 3.2 imply that $\mathcal{D}_0$ and $\mathcal{E}$ are closed and $\mu(\mathcal{D}_0) = \mu(\mathcal{E}) = 0$.

The following lemma states that the repeated application of the inverse mapping of f with PWC$^1\phi$ to a subset of measure zero results in a subset of measure zero in the case where acc$(\mathcal{R}_0) = \varnothing$. The proof shows that, for set $\mathcal{P}$ of positive measure, there exists an element χ of $\mathbb{R}^n \backslash (\mathcal{D}_0 \cup \mathcal{E})$ and a neighborhood around χ, U_χ, on which f' is nonsingular, such that $\mathcal{P} \cup U_\chi$ has positive measure, which is then used to show that $f(\mathcal{P} \cup U_\chi)$ has positive measure and thus $f(\mathcal{P})$ has positive measure. A contrapositive argument is then used to prove both results.

Lemma 3.5: Assume that ϕ is PWC^1 and acc$(\mathcal{R}_0) = \varnothing$. Then, the following hold:

i) Let $\mathcal{P} \subset \mathbb{R}^n$ be a measurable subset. If $\mu(f(\mathcal{P})) = 0$, then $\mu(\mathcal{P}) = 0$.

ii) Let $\mathcal{M} \subset \mathbb{R}^n$ be such that $\mu(\mathcal{M}) = 0$. For all $k \geq 1$, $\mu(f^{-k}(\mathcal{M})) = 0$.

Proof: To prove *i*), Proposition 3.4 implies that $\mathcal{D}_0$ and $\mathcal{E}$ are closed, and thus $\mathcal{U} \overset{\triangle}{=} \mathbb{R}^n \backslash (\mathcal{D}_0 \cup \mathcal{E})$ is open. Next, since $\mathcal{U} \cap (\mathcal{D}_0 \cup \mathcal{E}) = \varnothing$, it follows that f is C ^{1} on $\mathcal{U}$ and $f'(x)$ is nonsingular for all $x \in \mathcal{U}$. The inverse function theorem thus implies that, for all $x \in \mathcal{U}$, there exists an open neighborhood $U_x \subseteq \mathcal{U}$ of x and $V_x \subset \mathbb{R}^n$ of $f(x)$ such that $V_x = f(U_x)$, f is bijective on U_x, and f^{-1} is C^1 on V_x [54, Theorem 9.17], which implies that, for all $x \in \mathcal{U}$, $f : U_x \to V_x$ is a C^1 diffeomorphism. Note that $\cup_{x \in \mathcal{U}} U_x$ is an open covering of $\mathcal{U}$ and $\mathbb{R}^n$ is a Lindelö f space [55, p. 96]. Hence, there exists a countable subset $\mathcal{J} \subset \mathcal{U}$ such that $\mathcal{U} \subseteq \cup_{x \in \mathcal{J}} U_x$ and thus, for all $x \in \mathcal{J}$, $f : U_x \to V_x$ is a C^1 diffeomorphism.

Next, suppose that $\mu(\mathcal{P}) > 0$. Then, since $\mu(\mathcal{D}_0) = \mu(\mathcal{E}) = 0$ and $\mathcal{D}_0$, $\mathcal{E}$, and $\mathcal{U}$ are disjoint,

$$\begin{aligned}\mu(\mathcal{P}) &= \mu(\mathcal{P} \cap \mathcal{D}_0) + \mu(\mathcal{P} \cap \mathcal{E}) + \sum_{x \in \mathcal{J}} \mu(\mathcal{P} \cap \mathcal{U}_x) \\ &= \sum_{x \in \mathcal{J}} \mu(\mathcal{P} \cap \mathcal{U}_x),\end{aligned}$$

which implies that there exists $\chi \in \mathcal{J}$ such that $\mu(\mathcal{P} \cap U_\chi) > 0$. Since, for all $x \in U_\chi$, $f'(x)$ exists and is nonsingular, the change of variables theorem implies

$$\mu(f(\mathcal{P} \cap U_\chi)) = \int_{f(\mathcal{P} \cap \mathcal{U}_\chi)} d\mu(y) = \int_{\mathcal{P} \cap \mathcal{U}_\chi} |\det f'(x)| d\mu(x) > 0,$$

and thus $\mu(f(\mathcal{P})) > 0$. Hence, it follows from contraposition that $\mu(f(\mathcal{P})) = 0$ implies that $\mu(\mathcal{P}) = 0$.

To prove *ii*), since $f(f^{-1}(\mathcal{M})) \subseteq \mathcal{M}$, it follows that

$$0 \leq \mu(f(f^{-1}(\mathcal{M}))) \leq \mu(\mathcal{M}) = 0,$$

and thus

$$\mu(f(f^{-1}(\mathcal{M}))) = 0. \tag{16}$$

Hence, it follows from (16) and *ii*) with $\mathcal{P} = f^{-1}(\mathcal{M})$ that $\mu(f^{-1}(\mathcal{M})) = 0$. Finally, induction implies that, for all $k \geq 1$, $\mu(f^{-k}(\mathcal{M})) = 0$. □

C. NONCONVERGENCE OF THE SOLUTIONS OF THE LUR'E SYSTEM

Theorem 3.6 provides sufficient conditions under which the set of initial conditions for which the state trajectory of (1), (2) converges has measure zero, that is, for almost all initial conditions, the state trajectory of (1), (2) is nonconvergent. Note that, under the assumptions of Theorem 3.6, x_e is the unique equilibrium of (1), (2), $f'(x_e)$ is nonsingular and has at least one eigenvalue with magnitude greater than 1, and f' is nonsingular almost everywhere. The proof of Theorem 3.6 uses the stable manifold theorem to show that all convergent responses converge to a set of measure zero, after which Lemma 3.5*ii*) is used to complete the proof.

Theorem 3.6: If $I - A$ is nonsingular, $G(1) = 0$, $\phi(0) = 0$, ϕ is PWC^1, $\phi'(0)$ exists, acc$(\mathcal{R}_0) = \varnothing$, spr$(f'(x_e)) > 1$, and $f'(x_e)$ is nonsingular, then $\mu(\{x_0 \in \mathbb{R}^n : \lim_{k \to \infty} x_k$ exists$\}) = 0$.

Proof: Proposition 2.4*v*) implies that x_e is a fixed point of f. Since spr$(f'(x_e)) > 1$, define $\mathcal{X} \overset{\triangle}{=} x_e + \mathcal{Y}$, where $\mathcal{Y}$ is the proper subspace of $\mathbb{R}^n$ spanned by the generalized eigenvectors associated with the eigenvalues of $f'(x_e)$ whose magnitude is less than or equal to 1.

Since $f'(x_e)$ is nonsingular, the inverse function theorem implies that there exist open neighborhoods $U \subset \mathbb{R}^n$ of $x_e \in U$ and $V \subset \mathbb{R}^n$ of $f(x_e)$ such that $V = f(U)$, f is bijective on U, and f^{-1} is continuously differentiable on V [54, Theorem 9.17]. Then, the stable manifold theorem (Theorem III.7 in [56, pp. 65, 66]) implies that there exist a local f-invariant C^1 embedded disk $\mathcal{W} \subset \mathbb{R}^n$ and a ball $\mathcal{B}_{x_e}$ around x_e in an adapted norm such that $\mathcal{W}$ is tangent to $\mathcal{X}$ at $x_e, f(\mathcal{W}) \cap \mathcal{B}_{x_e} \subset \mathcal{W}, \mathcal{W}_{x_e} \overset{\triangle}{=} \cap_{p=0}^{\infty} f^{-p}(\mathcal{B}_{x_e}) \subset \mathcal{W}$, and, since spr$(f'(x_e)) > 1$, $\mathcal{W}$ has codimension of at least 1, and thus $\mu(\mathcal{W}) = 0$. Furthermore, since $\mathcal{W}_{x_e} \subset \mathcal{W}$, $\mu(\mathcal{W}_{x_e}) = 0$.

Next, let $\chi_0 \in \{x_0 \in \mathbb{R}^n : \lim_{k \to \infty} x_k = x_e\}$, and note that there exists $k_1 \geq 1$ such that, for all $k \geq k_1$, $f^k(\chi_0) \in \mathcal{B}_{x_e}$, which in turn implies that $f^{k_1}(\chi_0) \in \mathcal{W}_{x_e}$. This, in turn, implies that $\chi_0 \in \cup_{k=0}^{\infty} f^{-k}(\mathcal{W}_{x_e})$, and thus $\{x_0 \in \mathbb{R}^n : \lim_{k \to \infty} x_k = x_e\} \subseteq \cup_{k=0}^{\infty} f^{-k}(\mathcal{W}_{x_e})$. Hence, since $\mu(\mathcal{W}_{x_e}) = 0$, Lemma 3.5*ii* implies that

$$\begin{aligned}\mu(\{x_0 \in \mathbb{R}^n : \lim_{k \to \infty} x_k = x_e\}) &\leq \mu\left(\bigcup_{k=0}^{\infty} f^{-k}(\mathcal{W}_{x_e})\right) \\ &= \sum_{k=0}^{\infty} \mu(f^{-k}(\mathcal{W}_{x_e})) = 0,\end{aligned}$$

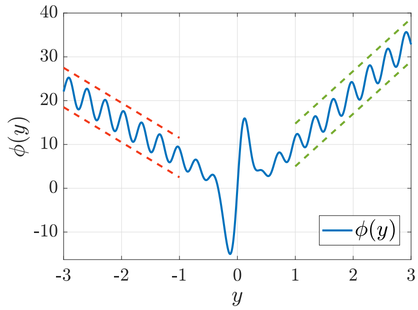

FIGURE 3. $\phi(y) = g(y) + h(\gamma y)$, where g and h are given by (17) and (18), and $\gamma = 4$, $\zeta = 3$, $\eta = 20$, $\mu = 0.125$, $s_l = -1$, $s_h = 1.5$. In this case, ϕ is affinely constrained by (α_l, α_h), where $\alpha_l = 2\gamma s_l = -8$ is the slope of the red, dashed line segments, and $\alpha_h = 2\gamma s_h = 12$ is the slope of the green, dashed line segments.

which, with Proposition 2.5*ii*)*b*), implies that

$$\mu(\{x_0 \in \mathbb{R}^n : \lim_{k\to\infty} x_k \text{ exists}\}) = 0. \qquad \square$$

Note that the assumptions of Theorem 3.6 do not imply that the solutions of (1), (2) are bounded. Definition 3.7 is used in Theorem 3.9 i) to state assumptions under which, for all initial conditions, the solution of (1), (2) is bounded.

D. AFFINELY CONSTRAINED FUNCTIONS

Definition 3.7: ϕ is *affinely constrained* if there exist $\alpha_l, \alpha_h, s_l, s_h \in \mathbb{R}$ and $\rho > 0$ such that $s_l < s_h$ and such that, for all $y \le s_l$, $|\phi(y) - \alpha_l y| < \rho$ and, for all $y \ge s_h$, $|\phi(y) - \alpha_h y| < \rho$. Furthermore, ϕ is *affinely constrained by* (α_l, α_h).

Note that, under Definition 3.7, every continuous and bounded nonlinearity ϕ is affinely constrained by (0, 0). The following example illustrates Definition 3.7.

Example 3.8: Let $\gamma, \zeta, \eta, \mu, s_l, s_h \in \mathbb{R}$, where $\mu \neq 0$, $s_l < 0 < s_h$, let $\phi(y) = g(y) + h(\gamma y)$, where $g, h : \mathbb{R} \to \mathbb{R}$ are given by

$$g(y) \triangleq \zeta \tanh(y)\sin(\eta y) + \frac{y}{\sqrt{2\pi\mu^3}} e^{\frac{-y^2}{2\mu^2}}, \tag{17}$$

$$h(y) \triangleq \begin{cases} s_l^2 + 2s_l(y - s_l), & y \le s_l, \\ y^2, & y \in (s_l, s_h), \\ s_h^2 + 2s_h(y - s_h), & y \ge s_h. \end{cases} \tag{18}$$

Since there exist $s > \max(\{-s_l,\ s_h\})$ and $\rho > |\zeta|$ such that, for all $y \notin [-s,\ s]$, $g(y) \in [-\rho, \rho]$, it follows that, for all $y < -s$, $|\phi(y) - 2\gamma s_l y| = |g(y) - s_l^2| < \rho + s_l^2$, and, for all $y > s$, $|\phi(y) - 2\gamma s_h y| = |g(y) - s_h^2| < \rho + s_h^2$. Hence, ϕ is affinely constrained by $(2\gamma s_l, 2\gamma s_h)$. Fig. 3 shows $\phi(y)$ for all $y \in [-3, 3]$ when $\gamma = 4$, $\zeta = 3$, $\eta = 20$, $\mu = 0.125$, $s_l = -1$, $s_h = 1.5$. In this case, ϕ is affinely constrained by $(-8, 12)$. $\diamond$

E. BOUNDEDNESS OF SOLUTIONS OF THE LUR'E SYSTEM

Theorem 3.9 i) below provides sufficient conditions under which (1), (2) is bounded for all initial conditions. Furthermore, under the assumptions of Theorem 3.6 and Theorem 3.9 i), Theorem 3.9 ii) shows that (1), (2) is self-excited. Note that the assumptions of Theorem 3.9 i) include the asymptotic stability of the linear dynamics of the Lur'e system and their feedback interconnection with linear mappings that correspond to the affine constraints that bound ϕ, as well as the positive definiteness of a set of matrices. The last assumption can be interpreted as the feasibility of the linear matrix inequality (LMI) shown in (21). The proof of Theorem 3.9 i) follows from rearranging the closed-loop dynamics as a switched linear system with bounded input, and using a common, quadratic Lyapunov function and input-to-state stability results.

Theorem 3.9: Assume $I - A$ is nonsingular, $\operatorname{spr}(A) < 1, G(1) = 0$, ϕ is continuous, and $\phi(0) = 0$, let $\alpha_l, \alpha_h \in \mathbb{R}$, assume ϕ is affinely constrained by (α_l, α_h), define $A_l \triangleq A + \alpha_l BC$ and $A_h \triangleq A + \alpha_h BC$, and assume that $\operatorname{spr}(A_l) < 1$ and $\operatorname{spr}(A_h) < 1$, and assume there exists positive-definite $P \in \mathbb{R}^{n\times n}$ such that $P - A^T PA$, $P - A_l^T PA_l$, and $P - A_h^T PA_h$ are positive definite. Then, the following hold:

i) For all $x_0 \in \mathbb{R}^n$, $(x_k)_{k=1}^\infty$ is bounded.

ii) Assume that ϕ is PWC[1] and differentiable at 0, $\operatorname{acc}(\mathcal{R}_0) = \varnothing$, $\operatorname{spr}(f'(x_e)) > 1$, and $f'(x_e)$ is nonsingular. Then, (1), (2) is self-excited.

Proof: To prove *i*), let $s_l < s_h$ and $\rho > 0$ be such that, for all $y \in (-\infty, s_l]$, $|\phi(y) - \alpha_l y| < \rho$, and, for all $y \in [s_h, \infty)$, $|\phi(y) - \alpha_h y| < \rho$. For all $k \ge 0$, (1) can be rewritten as

$$x_{k+1} = \begin{cases} (A + \alpha_l BC)x_k \\ +B(\phi(Cx_k) - \alpha_l Cx_k + v), & Cx_k \le s_l, \\ Ax_k + B(\phi(Cx_k) + v), & Cx_k \in (s_l, s_h), \\ (A + \alpha_h BC)x_k \\ +B(\phi(Cx_k) - \alpha_h Cx_k + v), & Cx_k \ge s_h. \end{cases} \tag{19}$$

Furthermore, defining

$$A_k \triangleq \begin{cases} A_l, & Cx_k \le s_l, \\ A, & Cx_k \in (s_l, s_h), \\ A_h, & Cx_k \ge s_h, \end{cases}$$

$$\nu_k \triangleq \begin{cases} \phi(Cx_k) - \alpha_l Cx_k + v, & Cx_k \le s_l, \\ \phi(Cx_k) + v, & Cx_k \in (s_l, s_h), \\ \phi(Cx_k) - \alpha_h Cx_k + v, & Cx_k \ge s_h, \end{cases}$$

(19) can be written as

$$x_{k+1} = A_k x_k + B\nu_k. \tag{20}$$

Since ϕ is continuous and affinely constrained by (α_l, α_h), $(\nu_k)_{k=0}^\infty$ is bounded. Next, define positive-definite

$$Q_l \triangleq P - A_l^T PA_l, \quad Q \triangleq P - A^T PA, \quad Q_h \triangleq P - A_h^T PA_h,$$

and $V(x) \triangleq x^T Px$. Then, for all $k \ge 0$, (20) implies

$V(x_{k+1}) - V(x_k)$

$$= \begin{cases} -x_k^{\mathrm{T}} Q_{\mathrm{l}} x_k + 2x_k^{\mathrm{T}} A_{\mathrm{l}}^{\mathrm{T}} P B v_k + v_k^{\mathrm{T}} B^{\mathrm{T}} P B v_k, & C x_k \le s_{\mathrm{l}}, \\ -x_k^{\mathrm{T}} Q x_k + 2x_k^{\mathrm{T}} A^{\mathrm{T}} P B v_k + v_k^{\mathrm{T}} B^{\mathrm{T}} P B v_k, & C x_k \in (s_{\mathrm{l}}, s_{\mathrm{h}}), \\ -x_k^{\mathrm{T}} Q_{\mathrm{h}} x_k + 2x_k^{\mathrm{T}} A_{\mathrm{h}}^{\mathrm{T}} P B v_k + v_k^{\mathrm{T}} B^{\mathrm{T}} P B v_k, & C x_k \ge s_{\mathrm{h}}. \end{cases}$$

Hence, for all $k \ge 0$,

$$V(x_{k+1}) - V(x_k) \le -\gamma(\|x_k\|) + \zeta(\|v_k\|),$$

where

$$\gamma(r) \triangleq \tfrac{1}{2} \min(\{\lambda_{\min}(Q_{\mathrm{l}}),\ \lambda_{\min}(Q),\ \lambda_{\min}(Q_{\mathrm{h}})\}) r^2,$$

$$\zeta(r) \triangleq \left[\max\left\{ \frac{2|A_{\mathrm{l}}^{\mathrm{T}} PB|^2}{\lambda_{\min}(Q_{\mathrm{l}})}, \frac{2|A^{\mathrm{T}} PB|^2}{\lambda_{\min}(Q)}, \frac{2|A_{\mathrm{h}}^{\mathrm{T}} PB|^2}{\lambda_{\min}(Q_{\mathrm{h}})} \right\} + |B^{\mathrm{T}} PB|^2 \right] r^2.$$

Since $\forall x \in \mathbb{R}^n$, $\lambda_{\min}(P)\|x\|_2^2 \le V(x) \le \lambda_{\max} x(P)\|x\|_2^2$, γ and ζ are continuous and strictly increasing, $\gamma(0) = \zeta(0) = 0$, and $\zeta(r) \to \infty$ as $r \to \infty$, Lemma 3.5 of [57] implies that (20) with input v is input-to-state stable. Since $(v_k)_{k=0}^{\infty}$ is bounded, it follows that, for all $x_0 \in \mathbb{R}^n$, $(x_k)_{k=1}^{\infty}$ is bounded. Finally, *ii*) follows from *i*) and Theorem 3.6. □

Note that Theorem 3.9 assumes that the LMI

$$\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & P - A^{\mathrm{T}} P A & 0 & 0 \\ 0 & 0 & P - A_{\mathrm{l}}^{\mathrm{T}} P A_{\mathrm{l}} & 0 \\ 0 & 0 & 0 & P - A_{\mathrm{h}}^{\mathrm{T}} P A_{\mathrm{h}} \end{bmatrix} > 0 \qquad (21)$$

is feasible, that is, there exists $P \in \mathbb{R}^{n \times n}$ such that the $4n \times 4n$ matrix in (21) is positive definite. Also, (21) holds with $P = I$ when $\|A\| < 1$, $\|A_{\mathrm{l}}\| < 1$, and $\|A_{\mathrm{h}}\| < 1$.

The following result is a corollary of Theorem 3.9*ii*) in the case where ϕ is bounded. In this case, ϕ is affinely constrained by $(0, 0)$, and thus $A_{\mathrm{l}} = A_{\mathrm{h}} = A$, which implies that (21) is feasible if and only if $\mathrm{spr}(A) < 1$.

Corollary 3.10: Assume $I - A$ is nonsingular, $G(1) = 0$, and $\phi(0) = 0$. Furthermore, assume $\mathrm{spr}\,(A) < 1$, ϕ is C^1 and bounded, $\mathrm{spr}(A + \phi'(0)BC) > 1$, and, for all $y \in \mathbb{R}$, $A + \phi'(y)BC$ is nonsingular. Then, (1), (2) is self-excited.

IV. NUMERICAL EXAMPLE

We now present an example to illustrate Theorem 3.9. In addition, a variation of this example shows that, if the feasibility of (21) is omitted from the hypotheses of Theorem 3.9, then the resulting statement is false. In particular, when (21) is not feasible, for a certain choice of ϕ, the response of (1), (2) exhibits divergence for various initial conditions, which shows that the system is not an SES. The feasibility of the LMI in (21) is determined by using the Matlab function *feasp*, which provides a feasible solution within numerical tolerance if and only if one exists.

Example 4.1: Let $v = 5$, let $\gamma, \mu, \eta, s_{\mathrm{l}}, s_{\mathrm{h}} \in \mathbb{R}$, where μ, η are nonzero and $s_{\mathrm{l}} < 0 < s_{\mathrm{h}}$, let ϕ be given by

$$\phi(y) = \begin{cases} s_{\mathrm{l}}(s_{\mathrm{l}}^2 + \gamma) + 3s_{\mathrm{l}}^2(y - s_{\mathrm{l}}) + \mu \sin(\eta(y - s_{\mathrm{l}})), & y \le s_{\mathrm{l}}, \\ y^3 + \gamma y, & y \in (s_{\mathrm{l}}, s_{\mathrm{h}}), \\ s_{\mathrm{l}}(s_{\mathrm{h}}^2 + \gamma) + 3s_{\mathrm{h}}^2(y - s_{\mathrm{h}}) + \mu \sin(\eta(y - s_{\mathrm{h}})), & y \ge s_{\mathrm{h}}, \end{cases} \qquad (22)$$

and let

$$G(\mathbf{z}) = \frac{\mathbf{z}^3 - 1.1\mathbf{z}^2 + 0.88\mathbf{z} - 0.78}{\mathbf{z}^4 + 0.1\mathbf{z}^3 + 0.77\mathbf{z}^2 - 10^{-3}\mathbf{z} - 7.8 \cdot 10^{-3}} \qquad (23)$$

with minimal realization

$$A = \begin{bmatrix} -0.1 & -0.77 & 10^{-3} & 7.8 \cdot 10^{-3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1.1 & 0.88 & -0.78 \end{bmatrix}.$$

Note that ϕ is not C^1 but it is PWC^1 with $\mathcal{S} = \{s_{\mathrm{l}}, s_{\mathrm{h}}\}$, for all $y \le s_{\mathrm{l}}$, $|\phi(y) - 3s_{\mathrm{l}}^2 y| = |\mu \sin(\eta(y - s_{\mathrm{l}})) - 2s_{\mathrm{l}}^3| \le |\mu| + 2|s_{\mathrm{l}}|^3$, and, for all $y \ge s_{\mathrm{h}}$, $|\phi(y) - 3s_{\mathrm{h}}^2 y| = |\mu \sin(\eta(y - s_{\mathrm{l}})) - 2s_{\mathrm{h}}^3| \le |\mu| + 2|s_{\mathrm{h}}|^3$. Hence, ϕ is affinely constrained by $(3s_{\mathrm{l}}^2, 3s_{\mathrm{h}}^2)$. Next, root-locus properties imply that $A + \phi'(y)BC$ is singular if and only if $\phi'(y) = 0.01$. For all $y \in \mathcal{R}$, ϕ' is given by

$$\phi'(y) = \begin{cases} 3s_{\mathrm{l}}^2 + \mu\eta \cos(\eta(y - s_{\mathrm{l}})), & y < s_{\mathrm{l}}, \\ 3y^2 + \gamma, & y \in (s_{\mathrm{l}}, s_{\mathrm{h}}), \\ 3s_{\mathrm{h}}^2 + \mu\eta \cos(\eta(y - s_{\mathrm{h}})), & y > s_{\mathrm{h}}, \end{cases}$$

which implies that

$$\mathcal{R}_0 \subset \{-\sqrt{|0.01 - \gamma|/3}, \sqrt{|0.01 - \gamma|/3}\}$$
$$\cup \{y \in \mathcal{R} : y < s_{\mathrm{l}} \text{ and } 0.01 - 3s_{\mathrm{l}}^2 = \mu\eta \cos(\eta(y - s_{\mathrm{l}}))\}$$
$$\cup \{y \in \mathcal{R} : y > s_{\mathrm{h}} \text{ and } 0.01 - 3s_{\mathrm{h}}^2 = \mu\eta \cos(\eta(y - s_{\mathrm{h}}))\},$$

which implies that $\mathrm{acc}(\mathcal{R}_0) = \varnothing$. Furthermore, $I - A$ is nonsingular, $\mathrm{spr}\,(A) < 1$, $G(1) = 0$, and $\phi(0) = 0$.

In particular, for $\gamma = 1.5, \mu = 0.1, \eta = 40, s_{\mathrm{l}} = -0.29$, $s_{\mathrm{h}} = 0.29$, it follows that $\mathrm{spr}(A + 3s_{\mathrm{l}}^2 BC) < 1$, $\mathrm{spr}(A + 3s_{\mathrm{h}}^2 BC) < 1$, and $\mathrm{spr}(f'(x_{\mathrm{e}})) > 1$. Furthermore, (21) is feasible with

$$P = \begin{bmatrix} 2.34 & -1.05 \cdot 10^{-1} & 1.14 & -1.13 \cdot 10^{-1} \\ -1.04 \cdot 10^{-1} & 1.74 & -1.07 \cdot 10^{-1} & 6.35 \cdot 10^{-1} \\ 1.14 & -1.07 \cdot 10^{-1} & 1.21 & -3.58 \cdot 10^{-2} \\ -1.13 \cdot 10^{-1} & 6.35 \cdot 10^{-1} & -3.58 \cdot 10^{-2} & 6.10 \cdot 10^{-1} \end{bmatrix}.$$

Hence, the assumptions of Theorem 3.9 hold. Accordingly, Fig. 4 shows that, for the indicated initial states, the response of (1), (2) is bounded and does not converge.

Furthermore, for $\gamma = 1.5, \mu = 0.1, \eta = 40, s_{\mathrm{l}} = -0.29$, $s_{\mathrm{h}} = 0.62$, it follows that $\mathrm{spr}(A + 3s_{\mathrm{l}}^2 BC) < 1$, $\mathrm{spr}(A + 3s_{\mathrm{h}}^2 BC) < 1$, and $\mathrm{spr}(f'(x_{\mathrm{e}})) > 1$. However, (21) is infeasible. Hence, the assumptions of Theorem 3.9 are not satisfied. Fig. 5 shows that the response of (1), (2) is unbounded for

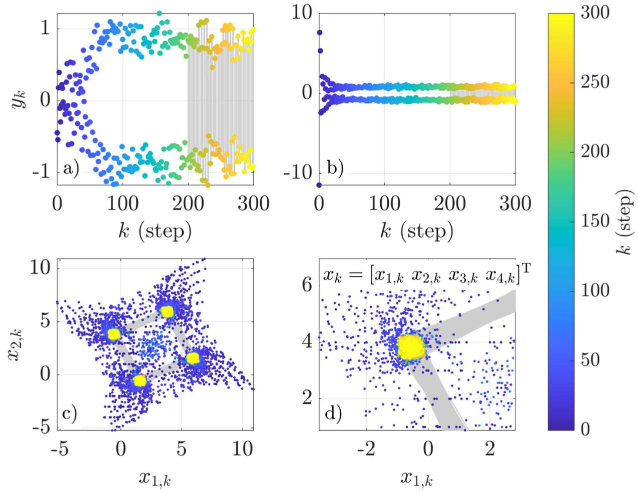

FIGURE 4. **Example 4.1: Response of (1), (2) for G given by (23), $v = 5$, ϕ is given by (22), and $\gamma = 1.5$, $\mu = 0.1$, $\eta = 40$, $s_l = -0.29$, $s_h = 0.29$. For all $k \in [0, 300]$, *a)* shows y_k for $x_0 = [2\ \ 4\ \ 4\ \ 2]^{\mathrm{T}}$. For all $k \in [0, 300]$, *b)* shows y_k for $x_0 = [-2\ \ 4\ \ -4\ \ 2]^{\mathrm{T}}$. For all $k \in [0, 300]$, *c)* shows x_k for all $x_0 \in \{-4, -3, \ldots, 4\} \times \{4\} \times \{-4, -3, \ldots, 4\} \times \{2\}$. *d)* is a magnified version of *c)*. For all $k \in [200, 300]$, the gray lines follow the trajectory from each initial state. .**

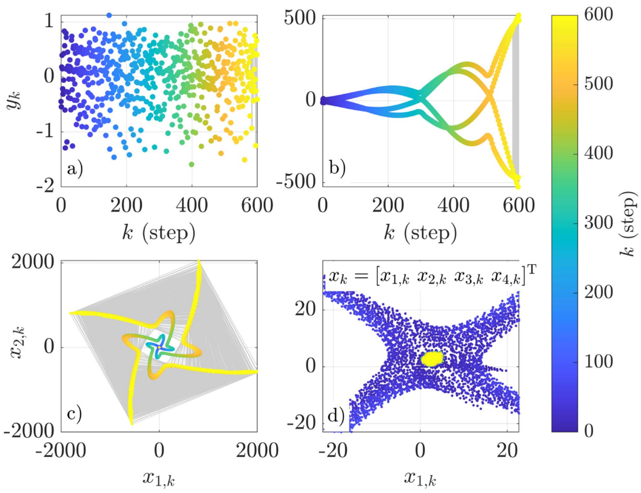

FIGURE 5. **Example 4.1: Response of (1), (2) for G given by (23), $v = 5$, ϕ is given by (22), and $\gamma = 1.5$, $\mu = 0.1$, $\eta = 40$, $s_l = -0.29$, $s_h = 0.62$. For all $k \in [0, 600]$, *a)* shows y_k for $x_0 = [2\ \ 4\ \ 4\ \ 2]^{\mathrm{T}}$. For all $k \in [0, 600]$, *b)* shows y_k for $x_0 = [-2\ \ 4\ \ -4\ \ 2]^{\mathrm{T}}$. For all $k \in [0, 600]$, *c)* shows x_k for all $x_0 \in \{-4, -3, \ldots, 4\} \times \{4\} \times \{-4, -3, \ldots, 4\} \times \{2\}$. *d)* is a magnified version of *c)*. For all $k \in [580, 600]$, the gray lines follow the trajectory from each initial state. .**

some initial states, which reinforces the statement that boundedness is not guaranteed for all initial states in the case where (21) is infeasible. ◊

V. CONCLUSIONS AND FUTURE WORK

This article considered discrete-time Lur'e systems whose response is self-excited, i.e., it is 1) bounded for all initial conditions, and 2) nonconvergent for almost all initial conditions. These models involve asymptotically stable linear dynamics with a washout filter connected in feedback with a piecewise-C^1 affinely constrained nonlinearity. Sufficient conditions on the growth rate of the nonlinearity were given under which the system is self-excited. Future work will focus on deriving analogous results for multiple-input, multiple-output (MIMO) systems.

APPENDIX

Proof of Proposition 2.4. To prove i), since $I - A$ is nonsingular, it follows that (7) and (10) are equivalent.

To prove $ii)a$), note that i) implies $Cx = C(I - A)^{-1} B(\phi(Cx) + v) = G(1)(\phi(Cx) + v)$. To prove necessity in $ii)b$), note that (10) implies $x = 0$. To prove sufficiency in $ii)b$), note that (10) implies $B(\phi(Cx) + v) = 0$. Since B is nonzero, it follows that $\phi(Cx) = -v$. To prove $ii)c$), note that, since $G(1) = 0$, it follows that $ii)a$) implies $Cx = G(1)(\phi(Cx) + v) = 0$. Furthermore, since $I - A$ is nonsingular, (10) implies that $x = (I - A)^{-1}B(\phi(0) + v)$ is the unique equilibrium of (1), (2). To prove $ii)d$), note that, since $Cx = 0$, it follows from $ii)a$) that $G(1)(\phi(0) + v) = 0$, which implies that either $G(1) = 0$ or $v = -\phi(0)$. To prove $ii)e$), note that, since $\phi(Cx) = 0$, (8) and (10) imply $x = x_{\mathrm{e}}$.

To prove iii), (9) implies $iii)b) \iff iii)c)$. Next, we show that $iii)a) \implies iii)b)$ and $iii)b) \implies iii)a)$. To prove $iii)a) \implies iii)b)$, note that (10) implies $x_{\mathrm{e}} = (I - A)^{-1}B(\phi(Cx_{\mathrm{e}}) + v) = (I - A)^{-1}Bv$, which implies $\phi(Cx_{\mathrm{e}}) = 0$. To prove $iii)b) \implies iii)a)$, note that $x_{\mathrm{e}} = (I - A)^{-1}Bv = (I - A)^{-1}B(\phi(Cx_{\mathrm{e}}) + v)$. Hence, i) implies x_{e} is an equilibrium.

iv) follows from iii) when $G(1) \neq 0$.

To prove v), we show $v)c) \implies v)b) \implies v)a) \implies v)c)$. $v)c) \implies v)b)$ is immediate. Next, since $G(1) = 0$, (9) implies $Cx_{\mathrm{e}} = G(1)v = 0$. Hence, iii) with $Cx_{\mathrm{e}} = 0$ implies that $\phi(Cx_{\mathrm{e}}) = \phi(0) = 0$, and thus $v)b) \implies v)a)$. Finally, since $G(1) = 0$, $ii)c$) implies that $x = (I - A)^{-1}B(\phi(0) + v)$ is the unique equilibrium of (1), (2). When $\phi(0) = 0$, $x = (I - A)^{-1}Bv = x_{\mathrm{e}}$ is the unique equilibrium of (1), (2), and thus $v)a) \implies v)c)$. □

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REFERENCES

[1] A. Jenkins, "Self-oscillation," *Phys. Rep.*, vol. 525, pp. 167–222, 2013.
[2] B. D. Coller and P. A. Chamara, "Structural non-linearities and the nature of the classic flutter instability," *J. Sound Vib.*, vol. 277, pp. 711–739, 2004.
[3] J. P. Epperlein, B. Bamieh, and K. J. Astrom, "Thermoacoustics and the Rijke tube: Experiments, identification, and modeling," *IEEE Control Syst. Mag.*, vol. 35, no. 2, pp. 57–77, Apr. 2015.
[4] A. Goldbeter and M. J. Berridge,*Biochemical Oscillations and Cellular Rhythms*. Cambridge, UK: Cambridge University Press, 1996.
[5] E. Livne, "Aircraft active flutter suppression: State of the art and technology maturation needs," *J. Aircr.*, vol. 55, pp. 410–52, 2018.
[6] A. P. Dowling and A. S. Morgans, "Feedback control of combustion oscillations," *Ann. Rev. Fluid Mech.*, vol. 37, pp. 151–182, 2005.

[7] C. Hang, K. Astrom, and Q. Wang, “Relay feedback auto-tuning of process controllers—A tutorial review,” *J. Proc. Control*, vol. 12, no. 1, pp. 143–162, 2002.
[8] S. M. Savaresi, R. R. Bitmead, and W. J. Dunstan, “Non-linear system identification using closed-loop data with no external excitation: The case of a lean combustion chamber,” *Int. J. Control*, vol. 74, pp. 1796–1806, 2001.
[9] J. A. Paredes, Y. Yang, and D. S. Bernstein, “Output-only identification of self-excited systems using discrete-time Lur’e models with application to a gas-turbine combustor,” *Int. J. Control*, vol. 97, no. 2, pp. 187–212, 2024.
[10] M. A. Aizerman, “On a problem concerning the stability ‘in the large’ of dynamical systems,” *Uspekhi Matematicheskikh Nauk*, vol. 4, pp. 187–188, 1949.
[11] V. Bragin, V. Vagaitsev, N. Kuznetsov, and G. Leonov, “Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua’s circuits,” *J. Comp. Syst. Sci. Int.*, vol. 50, no. 4, pp. 511–543, 2011.
[12] G. A. Leonov and N. V. Kuznetsov, “Hidden attractors in dynamical systems. from hidden oscillations in Hilbert–Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits,” *Int. J. Bifurcation Chaos*, vol. 23, no. 1, 2013, Art. no. 1330002.
[13] R. E. Kalman, “Lyapunov functions for the problem of Lur’e in automatic control,” *Proc. Nat. Acad. Sci.*, vol. 49, no. 2, pp. 201–205, 1963.
[14] M. A. Aizerman and F. R. Gantmakher, *Absolute Stability of Regulator Systems*. San Francisco, CA, USA: Holden-Day, 1964.
[15] V. Yakubovich, “A frequency theorem in control theory,” *Siberian Math. J.*, vol. 14, no. 2, pp. 265–289, 1973.
[16] X. Liu, J. Wang, Z. Duan, and L. Huang, “New absolute stability criteria for time-delay Lur’e systems with sector-bounded nonlinearity,” *Int. J. Robust Nonlinear Control*, vol. 20, no. 6, pp. 659–672, 2010.
[17] E. Sarkans and H. Logemann, “Input-to-state stability of Lur’e systems,” *Math. Control Signals Syst.*, vol. 27, no. 4, pp. 439–465, 2015.
[18] E. P. Popov, *The Dynamics of Automatic Control Systems*. Oxford, UK: Pergamon Press, 1962.
[19] V. Yakubovich, “Frequency-domain criteria for oscillation in nonlinear systems with one stationary nonlinear component,” *Siberian Math. J.*, vol. 14, no. 5, pp. 768–788, 1973.
[20] Y. Z. Tsypkin, *Relay Control Systems*. Cambridge, U.K.: Cambridge Univ. Press, 1984.
[21] E. A. Tomberg and V. A. Yakubovich, “Conditions for auto-oscillations in nonlinear systems,” *Siberian Math. J.*, vol. 30, no. 4, pp. 641–653, 1989.
[22] L. O. Chua and G.-N. Lin, “Canonical realization of Chua’s circuit family,” *IEEE Trans. Circuits Syst.*, vol. 37, no. 7, pp. 885–902, Jul. 1990.
[23] A. A. Alifov and K. V. Frolov, *Interaction of Nonlinear Oscillatory Systems with Energy Sources*. London, U.K.: Hemisphere, 1990.
[24] M. Arcak, M. Larsen, and P. Kokotović, “Boundedness without absolute stability in systems with stiffening nonlinearities,” *Eur. J. Control*, vol. 8, no. 3, pp. 243–250, 2002.
[25] M. Arcak and A. Teel, “Input-to-state stability for a class of Lurie systems,” *Automatica*, vol. 38, no. 11, pp. 1945–1949, 2002.
[26] R. Sepulchre and G.-B. Stan, “Feedback mechanisms for global oscillations in Lur’e systems,” *Syst. Control Lett.*, vol. 54, no. 8, pp. 809–818, 2005.
[27] D. V. Efimov and A. L. Fradkov, “Oscillatority of nonlinear systems with static feedback,” *SIAM J. Control Opt.*, vol. 48, pp. 618–40, 2009.
[28] W. M. Haddad and D. S. Bernstein, “Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and popov theorems and their application to robust stability. Part II: Discrete-time theory,” *Int. J. Robust Nonlinear Control*, vol. 4, no. 2, pp. 249–265, 1994.
[29] P. Park and S. W. Kim, “A revisited Tsypkin criterion for discrete-time nonlinear Lur’e systems with monotonic sector-restrictions,” *Automatica*, vol. 34, no. 11, pp. 1417–1420, 1998.
[30] N. S. Ahmad, W. P. Heath, and G. Li, “LMI-based stability criteria for discrete-time Lur’e systems with monotonic, sector-and slope-restricted nonlinearities,” *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 459–465, Feb. 2013.
[31] C. A. Gonzaga, M. Jungers, and J. Daafouz, “Stability analysis of discrete-time Lur’e systems,” *Automatica*, vol. 48, pp. 2277–2283, 2012.
[32] S. Townley, R. Rebarber, and B. Tenhumberg, “Feedback control systems analysis of density dependent population dynamics,” *Syst. Control Lett.*, vol. 61, no. 2, pp. 309–315, 2012.
[33] N. S. Ahmad, J. Carrasco, and W. P. Heath, “A less conservative LMI condition for stability of discrete-time systems with slope-restricted nonlinearities,” *IEEE Trans. Autom. Control*, vol. 60, no. 6, pp. 1692–1697, Jun. 2015.
[34] B. Y. Park, P. Park, and N. K. Kwon, “An improved stability criterion for discrete-time Lur’e systems with sector- and slope-restrictions,” *Automatica*, vol. 51, pp. 255–258, 2015.
[35] E. Sarkans and H. Logemann, “Input-to-state stability of discrete-time Lur’e systems,” *SIAM J. Control Optim.*, vol. 54, pp. 1739–1768, 2016.
[36] J. Zhang, J. Carrasco, and W. P. Heath, “Duality bounds for discrete-time Zames–Falb multipliers,” *IEEE Trans. Autom. Control*, vol. 67, no. 7, pp. 3521–3528, Jul. 2022.
[37] J. Zhang and J. Carrasco, “A Lyapunov-lurye functional parametrization of discrete-time zames-falb multipliers,” *IEEE Control Sys. Lett.*, vol. 6, pp. 259–264, 2022.
[38] A. L. Bertolin, R. C. Oliveira, G. Valmorbida, and P. L. Peres, “An LMI approach for stability analysis and output-feedback stabilization of discrete-time Lur’e systems using Zames-Falb multipliers,” *IEEE Control Syst. Lett.*, vol. 6, pp. 710–715, 2022.
[39] A. L. Bertolin, R. C. Oliveira, G. Valmorbida, and P. L. Peres, “Control design of uncertain discrete-time Lur’e systems with sector and slope bounded nonlinearities,” *Int. J. Robust Nonlinear Control*, vol. 32, no. 12, pp. 7001–7015, 2022.
[40] C. Guiver, R. Rebarber, and S. Townley, “Low-gain integral control for a class of discrete-time Lur’e systems with applications to sampled-data control,” *Int. J. Robust Nonlinear Control*, vol. 33, pp. 1410–1437, 2022.
[41] R. Drummond and G. Valmorbida, “Generalised Lyapunov functions for discrete- time Lurie systems with slope-restricted nonlinearities,” *IEEE Trans. Autom. Control*, vol. 68, no. 10, pp. 5966–5976, Oct. 2023.
[42] A. Kharitenko and C. W. Scherer, “On the exactness of a stability test for discrete-time Lur’e systems with slope-restricted nonlinearities,” *IEEE Trans. Autom. Control*, early access, Feb. 06, 2024, doi: 10.1109/TAC.2024.3362859.
[43] V. Rasvan, “Self-sustained oscillations in discrete-time nonlinear feedback systems,” in *Proc. Mediterranean Electrotech. Conf.*, 1998, pp. 563–565.
[44] M. B. D’Amico, J. L. Moiola, and E. E. Paolini, “Hopf bifurcation for maps: A frequency-domain approach,” *IEEE Trans. Circuits Syst. I: Fund. Theory Appl.*, vol. 49, no. 3, pp. 281–288, Mar. 2002.
[45] J. Paredes, S. A. Ul Islam, and D. S. Bernstein, “A time-delayed Lur’e model with biased self-excited oscillations,” in *Proc. Amer. Control Conf.*, 2020, pp. 2699–2704.
[46] P. Seiler and J. Carrasco, “Construction of periodic counterexamples to the discrete-time Kalman conjecture,” *IEEE Control Sys. Lett.*, vol. 5, no. 4, pp. 1291–1296, Oct. 2021.
[47] T. Van Pelt and D. S. Bernstein, “Nonlinear system identification using Hammerstein and nonlinear feedback models with piecewise linear static maps,” *Int. J. Control*, vol. 74, pp. 1807–1823, 2001.
[48] G. Dolanc and S. Strmcnik, “Identification of nonlinear systems using a piecewise-linear Hammerstein model,” *Syst. Control Lett.*, vol. 54, no. 2, pp. 145–158, 2005.
[49] J. Voros, “Parameter identification of wiener systems with multisegment piecewise-linear nonlinearities,” *Syst. Control Lett.*, vol. 56, no. 2, pp. 99–105, 2007.
[50] J. D. Lee, I. Panageas, G. Piliouras, M. Simchowitz, M. I. Jordan, and B. Recht, “First-order methods almost always avoid strict saddle points,” *Math. Prog.*, vol. 176, no. 1, pp. 311–337, 2019.
[51] D. Efimov, W. Perruquetti, and A. Shiriaev, “On existence of oscillations in hybrid systems,” *Non. Anal. Hybrid Syst.*, vol. 12, pp. 104–116, 2014.
[52] J. Wang, J. M. Avila, D. Efimov, A. Aleksandrov, and L. Fridman, “On existence of oscillations in Persidskii systems,” *IFAC Proc. Vols.*, vol. 53, no. 2, pp. 6305–6310, 2020.
[53] M. E. Gilmore, C. Guiver, and H. Logemann, “Stability and convergence properties of forced infinite-dimensional discrete-time Lur’e systems,” *Int. J. Control*, vol. 93, no. 12, pp. 3026–3049, 2020.
[54] W. Rudin, *Principles of Mathematical Analysis*, 2nd ed. New York, NY, USA: McGraw-Hill, 1964.

[55] G. J. O. Jameson, *Topology and Normed Spaces*. Hoboken, NJ, USA: Wiley, 1974.
[56] M. Shub, *Global Stability of Dynamical Systems*. Berlin, Germany: Springer, 1987.
[57] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.

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