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Initial Undershoot in Discrete-Time Input–Output Hammerstein Systems

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ABSTRACT This paper considers initial undershoot in the step response of discrete-time, input-output Hammerstein (DIH) systems, which have linear unforced dynamics and nonlinear zero dynamics (ZD). Initial undershoot occurs when the step response of a system moves initially in a direction that is opposite to the direction of the asymptotic response. For DIH systems, the paper investigates the relationship among the existence of initial undershoot, the step height, the height-dependent delay, and the stability of the ZD. For linear, time-invariant systems, the height-dependent delay specializes to the relative degree. The main result of the paper provides conditions under which, for all sufficiently small step heights, initial undershoot in the step response of a DIH system implies instability of the ZD. Several examples of DIH systems are presented to illustrate these results.

INDEX TERMS Nonlinear systems and control, stability of nonlinear systems, initial undershoot, nonlinear zero dynamics, discrete-time input-output Hammerstein system.

I. INTRODUCTION

One of the most striking phenomena in systems theory is initial undershoot, which occurs when a continuous-time linear system with an odd number of positive nonminimum-phase (NMP) zeros is subjected to a step input [9], [14], [17], [24]. NMP zeros limit achievable performance [8], [22], as can be seen from root locus analysis, where NMP zeros restrict the gain margin of the closed-loop system. In addition, NMP zeros are a serious impediment in adaptive control, where many adaptive control techniques are confined to systems whose zeros are minimum phase (MP) [10], [19]. The source of the difficulty is the inability to remove NMP zeros by inverting the plant, that is, by cancelling these zeros. Although cancellation of MP zeros entails no difficulty, cancellation of NMP zeros leads to internal divergence. These phenomena have motivated extensive studies of plant zeros [3], [16], [20], [21], [23].

Zeros of a linear system can be viewed as the “poles” of the zero dynamics (ZD) of the system, that is, the dynamics of the input that arise from constraining the output to be zero. This point of view extends the notion of zeros to nonlinear systems [13]. Unstable ZD in nonlinear systems are thus a natural extension of NMP zeros in linear systems. In fact, it is shown in [6] that, for a class of nonlinear, continuous-time systems with unstable ZD, the response due to a step input of sufficiently small height has initial undershoot.

Unstable ZD have also been studied within the context of sampled-data systems [11], [12], [18], [25]. In particular, analogous to the results of [6], it is shown in [25] for sampled-data systems that, for all step inputs of sufficiently small height, the response has initial undershoot.

The standard framework for analyzing ZD is normal forms, as can be seen from their central role in [13], within the context of continuous-time dynamics. Systems modeled by discrete-time normal forms have been studied in [2]. Neither of these works, however, studies initial undershoot.

The setting for the investigation of initial undershoot in nonlinear systems in the present paper is discrete-time, input-output Hammerstein (DIH) systems, which have linear unforced dynamics and nonlinear ZD. The contribution of the present paper is an investigation of the relationship between unstable ZD and initial undershoot. In particular, the main result given by Theorem 3 shows that instability of the ZD is a consequence of initial undershoot for all sufficiently small step heights. This result depends on the relationship among the existence of initial undershoot, the step height, the height-dependent delay, and the stability of the ZD. For linear,

TABLE 1. Definitions and results.

Definition/Result	Objective
Definition 1	Defines ZD of (1)
Definition 2	Defines zero-dynamics equilibrium (ZDE)
Proposition 1	Provides an expression for the limiting value of the output of (1) due to a step input
Definition 3	Defines initial undershoot in the output of (1) due to a step input
Lemma 1	Provides an expression for the first n outputs of (1) due to a step input
Definition 4	Defines height-dependent delay
Lemma 2	Provides equivalent expressions for infinite height-dependent delay
Proposition 2	Provides an expression for finite height-dependent delay
Theorem 1	Provides necessary and sufficient conditions for initial undershoot
Corollary 1	Provides necessary and/or sufficient conditions for initial undershoot in terms of input functions
Lemma 3	Provides sufficient conditions for height-dependent delay to be independent of $\overline{u}$
Corollary 2	Specializes Theorem 1 to DIH systems with identical nonlinearities
Corollary 3	Specializes Theorem 1 to linear systems
Theorem 2	Provides necessary and sufficient conditions for initial undershoot in linearized dynamics
Theorem 3	Provides a relationship between initial undershoot and unstable ZD

TABLE 2. Examples.

Example	Objective
Example 1	Illustrates infinite height-dependent delay for an affine input nonlinearity (explicit ZD)
Example 2	Illustrates infinite height-dependent delay for a trigonometric input nonlinearity (implicit ZD)
Example 3	Illustrates the dependence of $d(\overline{u})$ on $\overline{u}$ (implicit ZD)
Example 4	Illustrates Corollary 1 for a case with $d(\overline{u}) = n$ (implicit ZD)
Example 5	Illustrates the sufficient condition for initial undershoot in Theorem 1 (explicit ZD)
Example 6	Illustrates the necessary and sufficient condition for initial undershoot in Theorem 1 (explicit ZD)
Example 7	Illustrates Corollary 2 for a DIH system with deadzone nonlinearity (implicit ZD)
Example 8	Illustrates the sufficient condition for initial undershoot in the linearized DIH system in Theorem 2 (implicit ZD)
Example 9	Illustrates Theorem 2 with discontinuous d (implicit ZD)
Example 10	Illustrates the relationship between initial undershoot and unstable ZD in DIH systems in Theorem 3 (explicit ZD)

TABLE 3. Acronyms used in this paper.

Acronym	Meaning
DIH	discrete-time, input-output Hammerstein
MP	minimum phase
NMP	nonminimum phase
ZD	zero dynamics
ZDE	zero-dynamics equilibrium

time-invariant systems, the height-dependent delay specializes to the standard definition of relative degree. Theorem 3 can be viewed as a converse of the results given in [6] and [15].

The contents of the paper are as follows. Section II provides problem formulation, which involves a DIH system, the corresponding ZD, and preliminary results related to initial undershoot. Section III presents definitions, results, and examples related to initial undershoot in DIH systems. Section IV presents results and examples concerning the relationship between initial undershoot in the linearized dynamics and the DIH system, as well as the relationship between initial undershoot and the stability of the ZD. Table 1 describes the definitions and results in this paper, while Table 2 describes the illustrative examples. Table 3 provides acronyms used in the paper.

II. ZERO DYNAMICS

For all $k \geq 1$, consider the DIH system

$$y_k = \sum_{i=0}^{n-1} a_i y_{k-n+i} + \sum_{i=0}^{m} f_i(u_{k-n+i}), \tag{1}$$

which has linear unforced dynamics and nonlinear ZD, where $u_k \in \mathbb{R}$ is the input, $y_k \in \mathbb{R}$ is the output, n is a positive integer, m is a nonnegative integer such that $m < n, a_0, \ldots, a_{n-1} \in \mathbb{R}$, and, for all $i = 0, \ldots, m, f_i : \mathbb{R} \to \mathbb{R}$ is an *input function*. In a DIH system, linear system dynamics are cascaded with memoryless input nonlinearities, which arise in practice due to actuator nonlinearities, such as deadzone and saturation. The $m + 1$ input functions $f_0, \ldots, f_m$ in (1) represent the input nonlinearities of the DIH system. Hammerstein systems are widely studied in identification and control [1], [4], [7].

The initial conditions for (1) are the n prior values of y_k given by

$$y_{-n+1} = \cdots = y_{-1} = y_0 = 0. \tag{2}$$

The first possibly nonzero output is thus y_1. Finally, define the polynomial $D : \mathbb{C} \to \mathbb{R}$ by

$$D(z) \triangleq z^n - \sum_{i=0}^{n-1} a_i z^i. \tag{3}$$

With the convention $0^0 = 1$, it follows that $D(0) = -a_0$. We assume throughout that D is discrete-time asymptotically stable. It follows from [14, Lemma S1] that $D(1) > 0$.

Definition 1: The *zero dynamics (ZD)* of (1) are given by

$$\sum_{i=0}^{m} f_i(u_{k-n+i}) = 0, \tag{4}$$

where $k \geq 1$.

If f_m is bijective with inverse mapping f_m^{-1}, then, for all $k \geq m-n$, the ZD are given explicitly by

$$u_{k+1} = f_m^{-1}\left(\sum_{i=0}^{m-1} -f_i(u_{k-m+i+1})\right). \tag{5}$$

Otherwise, the ZD (4) are implicit.

Definition 2: $\overline{u}_{\text{eq}} \in \mathbb{R}$ is a *zero-dynamics equilibrium (ZDE)* if

$$\sum_{i=0}^{m} f_i(\overline{u}_{\text{eq}}) = 0. \tag{6}$$

To study initial undershoot in (1), we consider the *step input with height* $\overline{u} \in \mathbb{R}$ given by

$$u_k = \begin{cases} 0, & k \in \{-n+1, \ldots, -1\}, \\ \overline{u}, & k \geq 0. \end{cases} \tag{7}$$

The output of (1) with input (7) is the *step response of* (1). Note that the step input (7) changes from 0 to $\overline{u}$ at $k = 0$ preceded by $n-1$ zero values.

It follows from (1) and (7) that, for all $k \geq n$,

$$y_k = \sum_{i=0}^{n-1} a_i y_{k-n+i} + \sum_{i=0}^{m} f_i(\overline{u}). \tag{8}$$

Note that (8) is valid only for $k \geq n$. For all $k \in \{1, \ldots, n-1\}$, the prior value 0 of the step input (7) also affects y_k (see Lemma 1 in the next section). The initialization (2) of y_k with n prior zero values along with the $n-1$ prior zero values of u_k determine the startup transient of y_k, which will play a role in the definition of initial undershoot.

When the input u to (1) is the step (7), it follows that the second summation in (8) is a constant independent of k. This observation suggests that initial undershoot in the DIH system can be determined by considering an equivalent linear, time-invariant (LTI) system. This is not the case, however, since initial undershoot depends on the startup transient as well as the asymptotic output. In particular, Lemma 1 in Section III shows that the startup transient depends on the properties of the input functions.

Proposition 1: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then, $y_\infty \triangleq \lim_{k\to\infty} y_k$ exists and is given by

$$y_\infty = \frac{\sum_{i=0}^{m} f_i(\overline{u})}{D(1)}. \tag{9}$$

Proof: Defining $Y(z) \triangleq \mathcal{Z}\{y_k\}_{k=n}^{\infty} = \sum_{k=0}^{\infty} y_{k+n} z^{-k}$ and taking the Z-transform of both sides of (8) yields

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n-1} a_i y_{k+i} + \sum_{i=0}^{m} f_i(\overline{u})\right) z^{-k} \\ &= \sum_{i=0}^{n-1} a_i \sum_{k=0}^{\infty} y_{k+i} z^{-k} + \frac{z}{z-1}\sum_{i=0}^{m} f_i(\overline{u}). \end{aligned} \tag{10}$$

Note that, for all $i \in \{0, \ldots, n-1\}$,

$$\begin{aligned} \sum_{k=0}^{\infty} y_{k+i} z^{-k} &= z^{-n}\sum_{k=-n}^{-1} y_{k+n+i} z^{-k} + z^{-n}\sum_{k=0}^{\infty} y_{k+n+i} z^{-k} \\ &= z^{-n}\left(\sum_{k=-n}^{-1} y_{k+n+i} z^{-k} - \sum_{k=0}^{i-1} y_{k+n} z^{i-k}\right) \\ &\quad + z^{-n+i} Y(z). \end{aligned} \tag{11}$$

Substituting (11) into (10) yields

$$Y(z) = \frac{\sum_{i=0}^{n-1} a_i \left(\sum_{k=-n}^{-1} y_{k+n+i} z^{-k} - \sum_{k=0}^{i-1} y_{k+n} z^{i-k}\right)}{D(z)} + \frac{z^{n+1}\sum_{i=0}^{m} f_i(\overline{u})}{(z-1)D(z)}.$$

Since D is discrete-time asymptotically stable, the final value theorem [5, p. 40] implies

$$y_\infty = \lim_{z\to 1}(z-1)Y(z) = \frac{\sum_{i=0}^{m} f_i(\overline{u})}{D(1)}.$$

□

III. INITIAL UNDERSHOOT

The following definition expresses initial undershoot in terms of the startup transient and the asymptotic output.

Definition 3: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then y_k has *initial undershoot* if there exists $\kappa \geq 1$ such that

$$y_\kappa y_\infty < 0 \tag{12}$$

and such that, if $\kappa \geq 2$, then

$$y_1 = \cdots = y_{\kappa-1} = 0. \tag{13}$$

We use the following conventions: if $a > b$, then the interval $[a, b]$ is empty, and a sum over an empty set is zero.

Define $F : \{1, \ldots, n\} \times \mathbb{R} \to \mathbb{R}$ by

$$F(k, u) \triangleq \sum_{i=0}^{n-k-1} f_i(0) + \sum_{i=n-k}^{m} f_i(u). \tag{14}$$

The function F accounts for the startup transient due to the step input. As will be shown by Theorem 1, this function determines the presence of initial undershoot.

Lemma 1: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then, for all $k \in \{1, \ldots, n\}$,

$$y_k = \sum_{i=n-k+1}^{n-1} a_i y_{k-n+i} + F(k, \overline{u}). \tag{15}$$

Proof: Let $k \in \{1, \ldots, n\}$. Note that, for all $i \in \{0, \ldots, n-k\}$, it follows that $k - n + i \leq 0$. It thus follows from (2) that

$$\sum_{i=0}^{n-1} a_i y_{k-n+i} = \sum_{i=n-k+1}^{n-1} a_i y_{k-n+i}. \tag{16}$$

Furthermore, for all $i \in \{0, \ldots, n-k-1\}$, it follows that $k - n + i \leq -1$, and, for all $i \in \{n-k, \ldots, m\}$, it follows that $k - n + i \geq 0$. Hence, (7) implies that

$$\sum_{i=0}^{m} f_i(u_{k-n+i}) = F(k, \overline{u}). \tag{17}$$

Substituting (16) and (17) into (1) yields (15). □

Define $d : \mathbb{R} \to \{1, \ldots, n\} \cup \{\infty\}$ by

$$d(u) \triangleq \min\{k \in \{1, \ldots, n\} : F(k, u) \neq 0\}, \tag{18}$$

and define $d(u) \triangleq \infty$ when the set in (18) is empty. Since d takes integer values or may be ∞, it follows that, if d is not constant on $\mathbb{R}$, then it is discontinuous on $\mathbb{R}$.

Subsequent examples will show that, for the step response y_k of (1) with step-input height $\overline{u}$, $d(\overline{u})$ is the first step $k \geq 1$ at which y_k is nonzero. This observation motivates the following terminology.

Definition 4: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then, $d(\overline{u})$ is the *height-dependent delay* of (1).

If the DIH system (1) is LTI, then $d(\overline{u})$ is the relative degree; in this case, $d(\overline{u})$ is independent of $\overline{u}$ and satisfies $d(\overline{u}) \in \{1, \ldots, n\}$. For a DIH system, however, $d(\overline{u})$ may depend on $\overline{u}$ and may be infinite. The following result focuses on the case where $d(\overline{u}) = \infty$.

Lemma 2: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then, the following statements are equivalent:

i) For all $k \geq 1$, $y_k = 0$.
ii) For all $k \in \{1, \ldots, n\}$, $y_k = 0$.
iii) For all $k \in \{1, \ldots, n\}$, $F(k, \overline{u}) = 0$.
iv) $d(\overline{u}) = \infty$.

Proof: *i*) immediately implies *ii*). To prove *ii*) implies *iii*), suppose there exists $\kappa \in \{1, \ldots, n\}$ such that $F(\kappa, \overline{u}) \neq 0$. In the case $d(\overline{u}) = 1$, Lemma 1 implies that $y_1 = F(1, \overline{u}) \neq 0$. In the case $d(\overline{u}) \geq 2$, Lemma 1 implies that, for all $k \in \{1, \ldots, d(\overline{u}) - 1\}$, $y_k = 0$. Since, in addition, $F(d(\overline{u}), \overline{u}) \neq 0$, Lemma 1 implies that

$$\begin{aligned} y_{d(\overline{u})} &= a_{n-d(\overline{u})+1} y_1 + \cdots + a_{n-1} y_{d(\overline{u})-1} + F(d(\overline{u}), \overline{u}) \\ &= F(d(\overline{u}), \overline{u}) \\ &\neq 0. \end{aligned}$$

To prove *iii*) implies *i*), note that Lemma 1 implies that, for all $k \in \{1, \ldots, n\}, y_k = \sum_{i=0}^{n-1} a_i y_{k-n+i}$, which, using (2), implies that, for all $k \in \{1, \ldots, n\}$, $y_k = 0$. Since, in addition, (7) implies that, for all $k > n$, $\sum_{i=0}^{m} f_i(u_{k-n+i}) = \sum_{i=0}^{m} f_i(\overline{u}) = 0$, it follows from (1) that, for all $k > n$, $y_k = 0$. Therefore, for all $k \geq 1$, $y_k = 0$.

Finally, the equivalence of *iii*) and *iv*) follows from the definition of $d(\overline{u})$. □

We now consider the case where $d(\overline{u})$ is finite.

Proposition 2: Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Then, $d(\overline{u})$ is finite if and only if $d(\overline{u}) \in \{1, \ldots, n\}$. When these conditions are satisfied, the following statements hold:

i) $d(\overline{u}) = \min\{k \in \{1, \ldots, n\} : y_k \neq 0\}$.
ii) For all $k \in \{1, \ldots, d(\overline{u}) - 1\}$, $y_k = 0$.
iii) $y_{d(\overline{u})} = F(d(\overline{u}), \overline{u})$.

Proof: Necessity follows from (18). Sufficiency is immediate.

To prove *i*), since $d(\overline{u})$ is finite, (18) implies that, for all $k \in \{1, \ldots, d(\overline{u}) - 1\}$, $F(k, \overline{u}) = 0$, and that $F(d(\overline{u}), \overline{u}) \neq 0$. Since, in addition, $d(\overline{u}) \leq n$, Lemma 1 together with (2) implies that, for all $k \in \{1, \ldots, d(\overline{u}) - 1\}$, $y_k = 0$, and that $y_{d(\overline{u})} \neq 0$, which confirms *i*). *ii*) follows directly from *i*).

To prove *iii*), note that (1) implies that

$$y_{d(\overline{u})} = \sum_{i=0}^{n-1} a_i y_{d(\overline{u})-n+i} + \sum_{i=0}^{m} f_i(u_{d(\overline{u})-n+i}).$$

Since *ii*) and (2) imply that $y_{d(\overline{u})-n} = \cdots = y_{d(\overline{u})-1} = 0$, it follows that $y_{d(\overline{u})} = \sum_{i=0}^{m} f_i(u_{d(\overline{u})-n+i})$, which, combined with (7), confirms *iii*). □

Example 1: This example, which involves one input function, illustrates the case of infinite height-dependent delay for an affine input nonlinearity. Consider the DIH system

$$y_k = \tfrac{1}{2} y_{k-1} + u_{k-1} - 1, \tag{19}$$

which can be viewed as an LTI system with relative degree 1 driven by the constant input -1. Equivalently, (19) has the form of (1) with $n = 1$, $m = 0$, and $f_0(u) = u - 1$. Note that f_0 is affine and thus it is nonlinear. Note that, for all $k \geq -1$, the ZD (5) of (19) are given explicitly by

$$u_{k+1} = 1. \tag{20}$$

By Definition 4, the height-dependent delay is the minimum number of time steps required for the step input (7) to affect the output y_k of (19). Note that, for the step input $\overline{u} = 1$, the output y_k remains zero; otherwise, for the step input $\overline{u} \neq 1$, $y_1 = \overline{u} - 1 \neq 0$. Hence,

$$d(\overline{u}) = \begin{cases} 1, & \overline{u} \neq 1, \\ \infty, & \overline{u} = 1. \end{cases} \tag{21}$$

◇

Example 2: This example, which involves one input function, illustrates the case of infinite height-dependent delay for a trigonometric input nonlinearity. Consider the DIH system

$$y_k = \tfrac{1}{2}y_{k-1} + \sin u_{k-1}, \tag{22}$$

where $D(z) = z - \frac{1}{2}$, $n = 1$, and $m = 0$. Note that, for all $k \geq 1$, the implicit ZD (4) of (22) are given by

$$\sin u_{k-1} = 0. \tag{23}$$

First, let $\overline{u} = \pi$, and note that $f_0(\pi) = 0$, which implies that $d(\pi) = \infty$. In this case, Lemma 1 implies that $y_1 = f_0(\pi) = 0$, which, using (22), yields $y_2 = \frac{1}{2}y_1 + f_0(\pi) = 0$. Similarly, (22) implies that, for all $k > 2$, $y_k = f_0(\pi) = 0$. Therefore, for all $k \geq 1$, $y_k = 0$. On the other hand, letting $\overline{u} = \pi/2$, implies that $d(\pi/2) = 1$. Since, $f_0(\pi/2) = 1$, Lemma 1 implies that $y_1 = f_0(\pi/2) = 1 \neq 0$. Hence, ($\mathbb{Z}$ is the set of integers)

$$d(\overline{u}) = \begin{cases} \infty, & \overline{u} \in \{i\pi : i \in \mathbb{Z}\}, \\ 1, & \text{otherwise.} \end{cases} \tag{24}$$

◇

Example 3: This example, which involves two input functions, shows the dependence of $d(\overline{u})$ on $\overline{u}$. Consider the DIH system

$$y_k = \tfrac{1}{2}y_{k-1} + \sin(u_{k-1}) + \sin\left(\frac{u_{k-2}}{2}\right), \tag{25}$$

where $D(z) = z^2 - \frac{1}{2}z$, $n = 2$, and $m = 1$. Note that, for all $k \geq 1$, the implicit ZD (4) of (25) are given by

$$\sin(u_{k-1}) + \sin(\tfrac{u_{k-2}}{2}) = 0. \tag{26}$$

It follows from (14) that $F(1, \overline{u}) = \sin \overline{u}$ and $F(2, \overline{u}) = \sin \overline{u} + \sin \frac{\overline{u}}{2}$. Therefore, Lemma 1 implies that $y_1 = F(1, \overline{u}) = \sin \overline{u}$ and $y_2 = \frac{1}{2}y_1 + F(2, \overline{u}) = \frac{1}{2}y_1 + \sin \overline{u} + \sin \frac{\overline{u}}{2}$. It thus follows from Definition 4 that

$$d(\overline{u}) = \begin{cases} 1, & \overline{u} \in \mathbb{R}\backslash\{i\pi : i \in \mathbb{Z}\}, \\ 2, & \overline{u} \in \{(2i+1)\pi : i \in \mathbb{Z}\}, \\ \infty, & \overline{u} \in \{2i\pi : i \in \mathbb{Z}\}, \end{cases} \tag{27}$$

which implies that $d(\overline{u})$ is discontinuous at infinitely many points in $\mathbb{R}$. ◇

Define $h : \{u \in \mathbb{R} : d(u) \text{ is finite}\} \to \mathbb{R}$ by

$$h(u) \triangleq F(d(u), u) \sum_{i=0}^{m} f_i(u) \tag{28}$$

The following result gives necessary and sufficient conditions for initial undershoot in a DIH system. Note that this result is valid whether the ZD (4) are implicit or explicit.

Theorem 1: Let $\overline{u} \in \mathbb{R}$, assume that $d(\overline{u})$ is finite, and let y_k denote the step response of (1) with step input (7). Then, y_k has initial undershoot if and only if

$$h(\overline{u}) < 0. \tag{29}$$

Proof: To prove necessity, note that Definition 3 implies that there exists $\kappa \geq 1$ such that

$$y_\kappa y_\infty < 0 \tag{30}$$

and such that, in the case where $\kappa \geq 2$, $y_1 = \cdots = y_{\kappa-1} = 0$. It thus follows from Proposition 2 *ii*) and 2 *iii*) that $\kappa = d(\overline{u})$, which, using (30), implies that $y_{d(\overline{u})}y_\infty < 0$. Since, in addition, $D(1) > 0$, it follows that $y_{d(\overline{u})}y_\infty D(1) < 0$, which, combined with Fact 1, implies that $y_{d(\overline{u})} \sum_{i=0}^{m} f_i(\overline{u}) < 0$. It thus follows from Proposition 2 *iii*) that $F(d(\overline{u}), \overline{u}) \sum_{i=0}^{m} f_i(\overline{u}) < 0$, which, using (28), implies that $h(\overline{u}) < 0$.

To prove sufficiency, note that Fact 1 implies that $\sum_{i=0}^{m} f_i(\overline{u}) = y_\infty D(1)$, and Proposition 2 *iii*) implies that $F(d(\overline{u}), \overline{u}) = y_{d(\overline{u})}$. It thus follows from (28) that $h(\overline{u}) = y_{d(\overline{u})}y_\infty D(1) < 0$, which, since $D(1) > 0$, implies that $y_{d(\overline{u})}y_\infty < 0$. Since, in addition, Proposition 2 *ii*) implies that, for all $k \in \{1, \ldots, d(\overline{u}) - 1\}$, $y_k = 0$, Definition 3 with $\kappa = d(\overline{u})$ implies that y_k has initial undershoot. □

Substituting (14) into (28) yields

$$h(u) = \left(\sum_{i=0}^{n-d(u)-1} f_i(0) + \sum_{i=n-d(u)}^{m} f_i(u)\right) \sum_{i=0}^{m} f_i(u). \tag{31}$$

Defining

$$F_0(u) \triangleq \sum_{i=0}^{n-d(u)-1} f_i(0), \tag{32}$$

$$F_1(u) \triangleq \sum_{i=0}^{n-d(u)-1} f_i(u), \tag{33}$$

$$F_2(u) \triangleq \sum_{i=n-d(u)}^{m} f_i(u), \tag{34}$$

it follows that (31) can be written as

$$h(u) = [F_0(u) + F_2(u)][F_1(u) + F_2(u)]. \tag{35}$$

Using (31)–(35) yields the following immediate corollary of Theorem 1. This result emphasizes the role of the input functions in shaping the startup transient, which determines whether or not y_k has initial undershoot.

Corollary 1: Let $\overline{u} \in \mathbb{R}$, assume that $d(\overline{u})$ is finite, and let y_k denote the step response of (1) with step input (7). Then, (29) is satisfied if and only if

$$F_0(\overline{u})[F_1(\overline{u}) + F_2(\overline{u})] + F_1(\overline{u})F_2(\overline{u}) < -F_2(\overline{u})^2. \tag{36}$$

Furthermore, the following statements hold:

i) If $d(\overline{u}) = n$, then y_k does not have initial undershoot.
ii) If $d(\overline{u}) \leq n - 1$ and, for all $i = 0, \ldots, n - d(\overline{u}) - 1$, $f_i(0) = f_i(\overline{u})$, then y_k does not have initial undershoot.
iii) If $f_0(0) = \cdots = f_{n-d(\overline{u})-1}(0) = 0$, then y_k has initial undershoot if and only if $F_1(\overline{u})F_2(\overline{u}) < -F_2(\overline{u})^2$.

Example 4: This example illustrates Corollary 1 for the case where $d(\overline{u}) = n$. Consider the third-order DIH system

$$y_k = \sum_{i=0}^{2} a_i y_{k-3+i} + \sin(u_{k-3}) + \sin(2u_{k-2}) + \cos(u_{k-1}), \tag{37}$$

where the coefficients a_0, a_1, a_2 are such that D is asymptotically stable, $n = 3$, and $m = 2$. Note that, for all $k \geq 1$, the

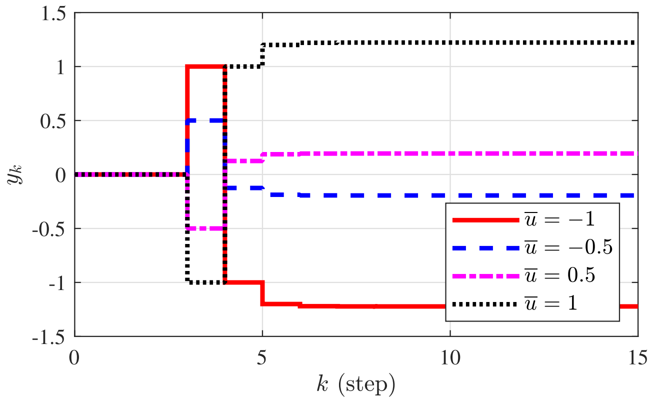

FIGURE 1. **Example 5. For all nonzero $\overline{u}$, $h(\overline{u}) < 0$, and the step response y_k of (43) has initial undershoot.**

implicit ZD (4) of (37) are given by

$$\sin(u_{k-3}) + \sin(2u_{k-2}) + \cos(u_{k-1}) = 0. \tag{38}$$

Consider the step input (7) with $\overline{u} = \frac{\pi}{2}$. Using (14) and Lemma 1 yields

$$y_1 = \sin 0 + \sin 0 + \cos\tfrac{\pi}{2} = 0, \tag{39}$$

$$y_2 = \sin 0 + \sin\pi + \cos\tfrac{\pi}{2} = 0, \tag{40}$$

$$y_3 = \sin\tfrac{\pi}{2} + \sin\pi + \cos\tfrac{\pi}{2} = 1 \neq 0. \tag{41}$$

Proposition 2 implies that $d(\overline{u}) = 3$. Therefore, *i*) of Corollary 1 implies that the step response of (37) with step input (7) and $\overline{u} = \frac{\pi}{2}$ does not have initial undershoot.

Alternatively, Proposition 1 implies that

$$y_\infty = \frac{\sin\frac{\pi}{2} + \sin\pi + \cos\frac{\pi}{2}}{D(1)} = \frac{1}{D(1)}. \tag{42}$$

Since $D(1) > 0$, it follows from (41) that $y_3 y_\infty > 0$. Thus, Definition 3 implies that the step response of (37) with step input (7) and $\overline{u} = \frac{\pi}{2}$ does not have initial undershoot. ⋄

Example 5: This example illustrates Theorem 1 by showing that (29) is a sufficient condition for initial undershoot. Consider the DIH system

$$y_k = 0.1y_{k-1} - u_{k-1} + \left(u_{k-2}^2 + 1.1\right) u_{k-2}, \tag{43}$$

where $D(z) = z^2 - 0.1z$, $n = 2$, and $m = 1$. Note that, for all $k \geq -1$, the ZD (5) of (43) are given explicitly by

$$u_{k+1} = \left(u_k^2 + 1.1\right) u_k. \tag{44}$$

It follows from (14) that $F(1, u) = -u$, which, together with (18), implies that, for all nonzero $\overline{u}$, $d(\overline{u}) = 1$. Therefore, (28) implies that, for all nonzero $\overline{u}$,

$$h(\overline{u}) = F(1, \overline{u}) \sum_{i=0}^{1} f_i(\overline{u}) = -\overline{u}^2 \left(\overline{u}^2 + 0.1\right) < 0. \tag{45}$$

In this case, Theorem 1 implies that, for all nonzero $\overline{u}$, the step response y_k of (43) has initial undershoot. Fig. 1 shows the step response of (43) with several heights. ⋄

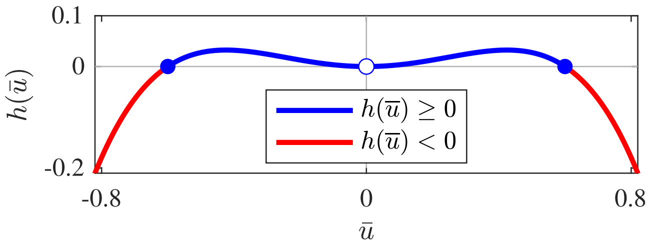

FIGURE 2. **Example 6. For all $\overline{u} \in [-0.6, 0) \cup (0, 0.6]$, $h(\overline{u}) \geq 0$, whereas, for all $\overline{u} < -0.6$ and all $\overline{u} > 0.6$, $h(\overline{u}) < 0$. The blue curve denotes step heights for which (46) does not have initial undershoot, whereas the red curve denotes step heights for which the step response y_k of (46) has initial undershoot. Since $d(0) = \infty$, $h(0)$ is not defined.**

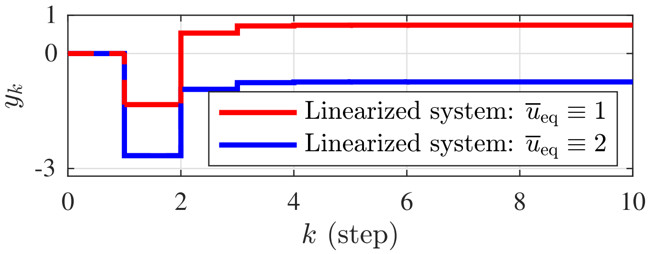

FIGURE 3. **Example 6. The step response y_k (red) of (46) with $\overline{u} = 1$ has initial undershoot, whereas, with $\overline{u} = 0.5$, the step response y_k (blue) of (46) does not have initial undershoot.**

Example 6: This example, which involves two input functions, illustrates Theorem 1 by showing that (29) is necessary and sufficient for initial undershoot. Consider the DIH system

$$y_k = 0.1y_{k-1} - u_{k-1} + \left(u_{k-2}^2 + 0.64\right) u_{k-2}, \tag{46}$$

where $D(z) = z^2 - 0.1z$, $n = 2$, $m = 1$, $f_0(u) = \left(u^2 + 0.64\right)u$, and $f_1(u) = -u$. Note that, for all $k \geq -1$, the ZD (5) of (46) are given explicitly by

$$u_{k+1} = \left(u_k^2 + 0.64\right) u_k. \tag{47}$$

It follows from (14) that $F(1, u) = -u$, which, together with (18), implies that, for all nonzero step heights $\overline{u}$, $d(\overline{u}) = 1$. Since $d(0) = \infty$, the case of zero step height is not pertinent to Theorem 1. Therefore, (28) implies that, for all nonzero $\overline{u}$,

$$h(\overline{u}) = F(1, \overline{u}) \sum_{i=0}^{1} f_i(\overline{u}) = \overline{u}^2 \left(0.36 - \overline{u}^2\right). \tag{48}$$

First, let $\overline{u} \in [-0.6, 0) \cup (0, 0.6]$, which implies that $0.36 - \overline{u}^2 \geq 0$. It thus follows from (48) that $h(\overline{u}) \geq 0$, as illustrated by the blue curve in Fig. 2 . In this case, Theorem 1 implies that y_k does not have initial undershoot. Next, let $\overline{u} \in \mathbb{R}\backslash[-0.6, 0.6]$, which implies that $0.36 - \overline{u}^2 < 0$. It thus follows from (48) that $h(\overline{u}) < 0$, as illustrated by the red curve in Fig. 2. In this case, Theorem 1 implies that y_k has initial undershoot. Fig. 3 shows the step response of (46) with $\overline{u} = 0.5$ and $\overline{u} = 1$. For $\overline{u} = 0.5$, y_k does not have initial undershoot, whereas, for $\overline{u} = 1$, y_k has initial undershoot. ⋄

The following corollary specializes Theorem 1 to the case of a DIH system with identical input nonlinearities. This result assumes that each input function f_i is of the form $b_i\sigma$, where $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies $\sigma(0) = 0$. The following lemma shows that this assumption implies that, for all step-input heights $\overline{u}$, $d(\overline{u}) = n - m$, and thus the height-dependent delay $d(\overline{u})$ is independent of $\overline{u}$. Note that σ is not assumed to be bijective, and thus the ZD (4) are not necessarily explicit.

Lemma 3: Assume that there exist $\sigma : \mathbb{R} \to \mathbb{R}$ satisfying $\sigma(0) = 0$ and $b_0, \dots, b_m \in \mathbb{R}$ such that b_m is nonzero. Let $\overline{u} \in \mathbb{R}$, and let y_k denote the step response of (1) with step input (7). Assume that $\sigma(\overline{u}) \neq 0$, and assume that, for all $i = 1, \dots, m$, $f_i(\overline{u}) = b_i\sigma(\overline{u})$. Then, $d(\overline{u}) = n - m$.

Proof: Since $\sigma(0) = 0$, it follows from (14) that

$$F(k, \overline{u}) = \sigma(\overline{u}) \sum_{i=n-k}^{m} b_i. \tag{49}$$

Since, in addition, b_m and $\sigma(\overline{u})$ are nonzero, it follows from (18) that $d(\overline{u}) = n - m$. □

Note that Lemma 3 applies trivially to Example 1 since (19) involves only one input function. On the other hand, Lemma 3 does not apply to Example 3 since $f_1(u) = \sin(u)$ and $f_0(u) = \sin(\frac{u}{2})$.

Corollary 2: Let $\sigma : \mathbb{R} \to \mathbb{R}$, assume that $\sigma(0) = 0$, let $\overline{u} \in \mathbb{R}$, assume that $\sigma(\overline{u}) \neq 0$, let $b_0, \dots, b_m \in \mathbb{R}$, where b_m is nonzero, and assume that, for all $i = 0, \dots, m, f_i(u) \triangleq b_i\sigma(u)$. Then, the step response y_k of (1) with step input $\overline{u} \in \mathbb{R}$ has initial undershoot if and only if $N(z) \triangleq \sum_{i=0}^{m} \frac{b_i}{b_m} z^i$ has an odd number of real roots greater than 1.

Proof: Since $\sigma(0) = 0$, $\sigma(\overline{u}) \neq 0$, and $b_m \neq 0$, Lemma 3 implies that $d(\overline{u}) = n - m$. It thus follows from (49) that $F(d(\overline{u}), \overline{u}) = b_m\sigma(\overline{u})$, which, together with (28), implies that

$$h(\overline{u}) = b_m[\sigma(\overline{u})]^2 \sum_{i=0}^{m} b_i = [b_m\sigma(\overline{u})]^2 N(1). \tag{50}$$

To prove necessity, since Theorem 1 implies that $h(\overline{u}) < 0$, it follows from (50) that $N(1) < 0$. It thus follows from [14, Lemma S1] that $N(z)$ has an odd number of real roots greater than 1.

To prove sufficiency, note that [14, Lemma S1] implies that $N(1) < 0$, which, combined with (50), implies that $h(\overline{u}) < 0$. It thus follows from Theorem 1 that the step response y_k of (1) has initial undershoot. □

Example 7: This example illustrates Corollary 2 for a DIH system with deadzone nonlinearity, which often occurs in practice. Since the deadzone nonlinearity is not injective, it follows that the ZD (4) are implicit. Consider the third-order DIH system

$$y_k = \sum_{i=0}^{2} a_i y_{k-3+i} + \sum_{i=0}^{2} f_i(u_{k-3+i}), \tag{51}$$

where $n = 3$, $m = 2$, and, for all $i \in \{0, 1, 2\}$, $f_i(u) = b_i\sigma(u)$, where $b_0 = b_1 = -b_2 = \frac{1}{2}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ is the deadzone function

$$\sigma(u) = \begin{cases} s_l(u - t_l), & u < t_l, \\ 0, & t_l \le u \le t_r, \\ s_r(u - t_r), & u > t_r, \end{cases} \tag{52}$$

where the deadzone parameters are the left and right slopes $s_l > 0$ and $s_r > 0$, and the left and right thresholds $t_l < 0$ and $t_r > 0$. The coefficients a_0, a_1, a_2 are such that D is asymptotically stable. Note that, for all $k \ge 1$, the implicit ZD (4) of (51) are given by

$$\sigma(u_{k-3}) + \sigma(u_{k-2}) - \sigma(u_{k-1}) = 0. \tag{53}$$

Let $\overline{u} \in (-\infty, t_l) \cup (t_r, \infty)$ be the height of the step input. It thus follows from (52) that $\sigma(\overline{u}) \neq 0$. Since $\sigma(0) = 0$, $b_2 \neq 0$, and $N(z) = z^2 - z - 1$ has exactly one real root greater than 1, Corollary 2 implies that the step response y_k of (51) has initial undershoot.

Fig. 4 illustrates initial undershoot in the step response of (51) with $\overline{u} = -4, -2, 3, 5$, where $a_0 = \frac{3}{10}$, $a_1 = -\frac{1}{2}$, $a_2 = 1$, and the deadzone parameters are $s_r = s_l = 2$ and $t_r = -t_l = 1$. ◇

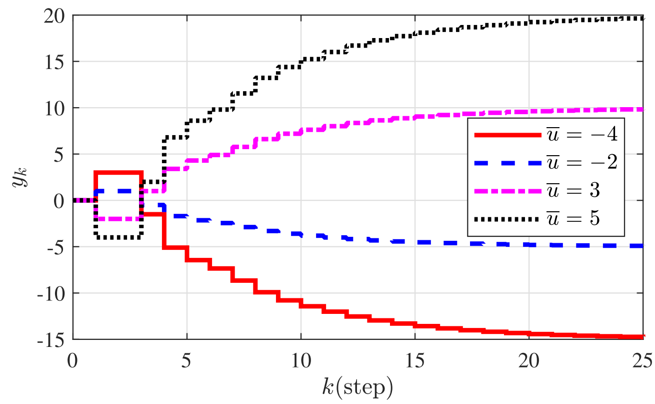

FIGURE 4. Example 7. For all step-input heights $\overline{u} \in \{-4, -2, 3, 5\}$, the step response of the DIH system (51) with deadzone nonlinearity has initial undershoot.

The following result specializes Corollary 2 to the case of linear dynamics. This result, which is given by [17] and [14, Theorem 5], presents a necessary and sufficient condition for the step response of a discrete-time linear system to have initial undershoot.

Corollary 3: Let $b_0, \dots, b_m \in \mathbb{R}$, assume that b_m is nonzero, assume that, for all $i = 0, \dots, m, f_i(u) = b_i u$, let $\overline{u} \in \mathbb{R}$, and assume that $\overline{u}$ is nonzero. Then, the step response y_k of (1) with step input (7) has initial undershoot if and only if $N(z) \triangleq \sum_{i=0}^{m} \frac{b_i}{b_m} z^i$ has an odd number of real roots greater than 1.

IV. INITIAL UNDERSHOOT AND ZERO DYNAMICS

In this section, we linearize the nonlinear dynamics (1), and we relate initial undershoot in the step response of the linearized dynamics to initial undershoot in the step response of (1). Furthermore, we investigate the relationship between initial undershoot and unstable ZD.

Linearizing (1) about $y_k \equiv 0$ and the ZDE $u \equiv \overline{u}_{\rm eq}$ yields the linear dynamics

$$\delta y_k = \sum_{i=0}^{n-1} a_i \delta y_{k-n+i} + \sum_{i=0}^{m} b_i \delta u_{k-n+i}, \tag{54}$$

where, for all $i = 0, \ldots, m$,

$$b_i \triangleq f_i'(\overline{u}_{\rm eq}), \tag{55}$$

assuming that the indicated derivatives exist.

The following result provides a necessary and sufficient condition for the existence of initial undershoot in the step response of the discrete-time linearized system (54).

Theorem 2: Let $\overline{u}_{\rm eq} \in \mathbb{R}$, assume that $\overline{u}_{\rm eq}$ is a ZDE, and assume that the following conditions hold:

i) For all $i = 1, \ldots, m$, $f_i'(\overline{u}_{\rm eq})$ exists.

ii) $d(\overline{u}_{\rm eq})$ is finite and constant in a neighborhood of $\overline{u}_{\rm eq}$.

iii) $m_0 \triangleq \max\{i : b_i \neq 0\}$ exists.

Then, the step response δy_k of the linearized system (54) has initial undershoot for all nonzero step inputs if and only if

$$b_{m_0} h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) < 0, \tag{56}$$

where h and F are defined by (28) and (14), respectively.

Proof: It follows from *i*) that $f_i'(\overline{u}_{\rm eq})$ exists, and it follows from *ii*) and (14) that $\mathrm{d}F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq})/\mathrm{d}u$ exists. Hence, using (6) and (55), it follows from (28) that

$$\begin{aligned} h'(\overline{u}_{\rm eq}) &= F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) \sum_{i=0}^{m} f_i'(\overline{u}_{\rm eq}) \\ &= F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) \sum_{i=0}^{m_0} b_i \\ &= b_{m_0} F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) N(1), \end{aligned} \tag{57}$$

where $N(z) \triangleq \sum_{i=0}^{m_0} \frac{b_i}{b_{m_0}} z^i$. Note that *ii*) implies that $F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) \neq 0$, and *iii*) implies that $b_{m_0} \neq 0$. Multiplying both sides of (57) by $b_{m_0} F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq})$ yields

$$b_{m_0} h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) = b_{m_0}^2 (F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}))^2 N(1). \tag{58}$$

To prove necessity, note that (54) has the form of (1), where, for all $i = 0, \ldots, m_0$, $f_i(\delta \overline{u}) = b_i \delta \overline{u}$. It thus follows from Corollary 3 that $N(z)$ has an odd number of real roots greater than 1. Therefore, [14, Lemma S1] implies that $N(1) < 0$, which, together with (58), implies that $b_{m_0} h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) < 0$.

To prove sufficiency, note that (58) implies that $N(1) < 0$, which, using [14, Lemma S1], implies that $N(z)$ has an odd number of real roots greater than 1. Therefore, since (54) has the form of (1), where, for all $i = 0, \ldots, m_0$, $f_i(\delta \overline{u}) = b_i \delta \overline{u}$, it follows from Corollary 3 that δy_k has initial undershoot. □

Example 8: This example illustrates Theorem 2 by showing that (56) is a sufficient condition for initial undershoot in the linearized system (54). Consider the DIH system

$$y_k = 0.1 y_{k-1} - \tfrac{2}{3} u_{k-1}^2 + 2 u_{k-2} - \tfrac{4}{3}, \tag{59}$$

where $D(z) = z^2 - 0.1z$, $n = 2$, and $m = 1$. Note that, for all $k \geq 1$, the implicit ZD (4) of (59) are given by

$$-\tfrac{2}{3} u_{k-1}^2 + 2 u_{k-2} - \tfrac{4}{3} = 0. \tag{60}$$

Moreover, $\overline{u}_{\rm eq}$ satisfies (6) if and only if $\sum_{i=0}^{1} f_i(\overline{u}_{\rm eq}) = -\frac{2}{3}\overline{u}_{\rm eq}^2 + 2\overline{u}_{\rm eq} - \frac{4}{3} = 0$. Hence, $\overline{u}_{\rm eq} = 1$ or $\overline{u}_{\rm eq} = 2$. It follows from (14) that $F(1, u) = -\frac{2}{3}u^2 - \frac{4}{3}$, which, together with (18), implies that, for all $\overline{u} \in \mathbb{R}$, $d(\overline{u}) = 1$, and thus *ii*) of Theorem 2 is satisfied. Therefore, (28) implies that, for all $\overline{u} \in \mathbb{R}$,

$$\begin{aligned} h(\overline{u}) &= F(1, \overline{u}) \sum_{i=0}^{1} f_i(\overline{u}) \\ &= \left(\frac{2}{3}\overline{u}^2 + \tfrac{4}{3}\right)\left(\frac{2}{3}\overline{u}^2 - 2\overline{u} + \tfrac{4}{3}\right), \end{aligned} \tag{61}$$

and, since *i*) of Theorem 2 holds, differentiating (61) with respect to $\overline{u}$ yields

$$h'(\overline{u}) = \tfrac{16}{9}\overline{u}_{\rm eq}^3 - 4\overline{u}_{\rm eq}^2 + \tfrac{32}{9}\overline{u}_{\rm eq} - \tfrac{8}{3}. \tag{62}$$

Note that (55) implies that $b_1 = -\frac{4}{3}\overline{u}_{\rm eq}$, and thus *iv*) of Theorem 2 is satisfied with $m_0 = 1$. It thus follows from (62) that

$$\begin{aligned} & b_1 h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) \\ &= \tfrac{4}{3}\overline{u}_{\rm eq}\left(\frac{16}{9}\overline{u}_{\rm eq}^3 - 4\overline{u}_{\rm eq}^2 + \tfrac{32}{9}\overline{u}_{\rm eq} - \tfrac{8}{3}\right)\left(\frac{2}{3}\overline{u}_{\rm eq}^2 + \tfrac{4}{3}\right). \end{aligned} \tag{63}$$

For $\overline{u}_{\rm eq} = 1$, it follows from (63) that

$$b_{m_0} h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) = -3.56 < 0,$$

which, using Theorem 2, implies that (59) linearized about $y_k \equiv 0$ and $\overline{u}_{\rm eq} \equiv 1$ has initial undershoot.

For $\overline{u}_{\rm eq} = 2$, it follows from (63) that

$$b_{m_0} h'(\overline{u}_{\rm eq}) F(d(\overline{u}_{\rm eq}), \overline{u}_{\rm eq}) = 28.44 > 0,$$

which, using Theorem 2, implies that (59) linearized about $y_k \equiv 0$ and $\overline{u}_{\rm eq} \equiv 2$ does not have initial undershoot. Fig. 5 shows the step response of the system linearized about $y_k \equiv 0$ with $\overline{u}_{\rm eq} \equiv 1$, as well as the step response of the system linearized about $y_k \equiv 0$ with $\overline{u}_{\rm eq} \equiv 2$. ◇

Example 9: This example shows that the conclusions of Theorem 2 may remain valid in the case where d is discontinuous at $\overline{u}_{\rm eq}$ and thus is not constant in a neighborhood of $\overline{u}_{\rm eq}$. Consider the DIH system

$$\begin{aligned} y_k = {} & 0.1 y_{k-1} + \sin(2u_{k-1}) - \sin(u_{k-2}) - 2\sin(2u_{k-2}) \\ & + \sin(u_{k-3}), \end{aligned} \tag{64}$$

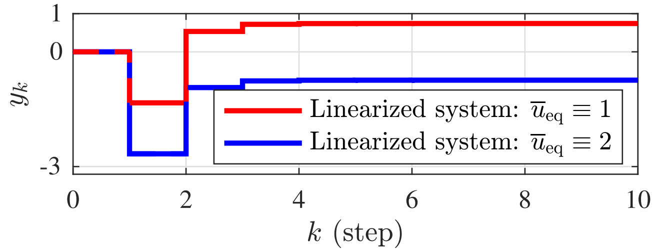

FIGURE 5. **Example 8. For the step-input height $\overline{u} = 1$, the step response (red) of the DIH system (59) linearized about $y_k \equiv 0$ and $\overline{u}_{\rm eq} \equiv 1$ has initial undershoot, whereas the step response (blue) of the DIH system (59) linearized about $y_k \equiv 0$ and $\overline{u}_{\rm eq} \equiv 2$ does not have initial undershoot.**

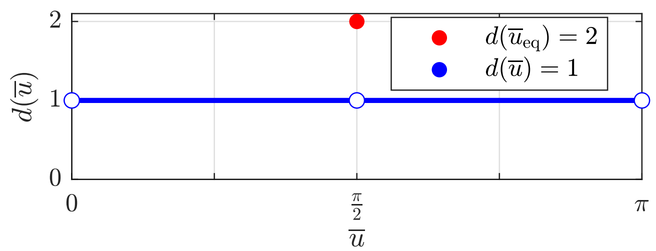

FIGURE 6. **Example 9. For all $\overline{u} = k\pi$, where k is an integer, it follows that $d(\overline{u}) = \infty$..**

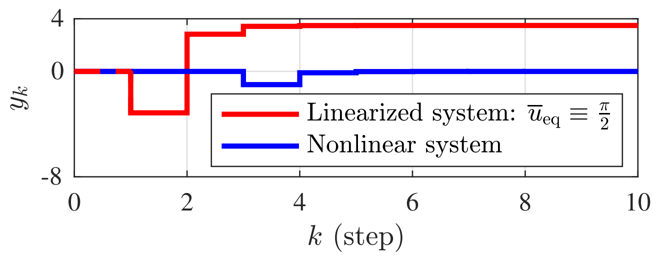

FIGURE 7. **Example 9. The step response (blue) of the DIH system (64) with $\overline{u} = \frac{\pi}{2}$ does not have initial undershoot. However, the step response (red) of the linearized DIH system (66) linearized about $y_k \equiv 0$ and $\overline{u}_{\rm eq} \equiv \frac{\pi}{2}$ with step input $\overline{u} = \frac{\pi}{2}$ has initial undershoot.**

where $D(z) = z^3 - 0.1z^2$, $n = 3$, and $m = 2$. Note that, for all $k \geq 1$, the implicit ZD (4) of (64) are given by

$$\sin(2u_{k-1}) - \sin(u_{k-2}) - 2\sin(2u_{k-2}) + \sin(u_{k-3}) = 0. \tag{65}$$

Moreover, $\overline{u}_{\rm eq}$ satisfies (6), which implies that $\sum_{i=0}^{2} f_i(\overline{u}_{\rm eq}) = -\sin(2\overline{u}_{\rm eq}) = 0$. Since, for all $k \in \mathbb{Z}$, $d(k\pi) = \infty$ and $d(\frac{2k-1}{2}\pi) = 2$, it follows that $\overline{u}_{\rm eq} = \frac{2k-1}{2}\pi$, and $d(\overline{u}_{\rm eq}) = 2$. Let $\overline{u}_{\rm eq} \equiv \frac{\pi}{2}$. Fig. 6 shows $d(\overline{u})$ of (64).

Next, linearizing (64) about $y_k \equiv 0$ and $\overline{u}_{\rm eq}$ yields

$$\delta y_k = 0.1\delta y_{k-1} - 2\delta u_{k-1} + 4\delta u_{k-2}, \tag{66}$$

which has the transfer function $G(z) = -\frac{2N(z)}{z(z-0.1)}$, where $N(z) = z - 2$ has exactly one real zero greater than 1. Corollary 3 with $m = 2$, $b_0 = 0$, $b_1 = 4$, and $b_2 = -2$ thus implies that (66) has initial undershoot. Fig. 7 shows the step response of (64) and (66) with $\overline{u} = \frac{\pi}{2}$. ◇

The following result, which is a key contribution of this paper, concerns the relationship between initial undershoot and unstable ZD in DIH systems. Note that *v*) implies that the ZD are locally explicit.

Theorem 3: Let $\overline{u}_{\rm eq}$ be a ZDE, define $b_0, \ldots, b_m$ by (55), and assume that the following conditions hold:

i) For all $i = 0, \ldots, m$, f_i is C^2.
ii) $f_m(\overline{u}_{\rm eq}) = 0$, and, for all $i = 0, \ldots, m$, $f_i(0) = 0$.
iii) $b_m \neq 0$ and $\overline{b} \triangleq \sum_{i=0}^{m} b_i \neq 0$.
iv) There exists $\varepsilon > 0$ such that, for all $\delta u \in (-\varepsilon, 0) \cup (0, \varepsilon)$, $h(\overline{u}_{\rm eq} + \delta u) < 0$.
v) f_m is bijective in a neighborhood of $\overline{u}_{\rm eq}$.

Then, $\overline{u}_{\rm eq}$ is an unstable ZDE.

Proof: For all $i = 0, \ldots, m$, *i*) implies that there exists $\varepsilon_0 \in (0, \varepsilon)$, where ε is given by *iv*), such that, for all $\delta u \in (-\varepsilon_0, \varepsilon_0)$,

$$f_i(\overline{u}_{\rm eq} + \delta u) = f_i(\overline{u}_{\rm eq}) + b_i\delta u + r_i(\delta u), \tag{67}$$

where $r_i(\delta u) = O(\delta u^2)$. Next, since, for all $i = 0, \ldots, m$, $f_i(0) = 0$, it follows from (14) that, for all $\overline{u} \in \mathbb{R}$,

$$F(k, \overline{u}) = \begin{cases} 0, & k \in \{1, \ldots, n-m-1\}, \\ \sum_{i=n-k}^{m} f_i(\overline{u}), & k \in \{n-m, \ldots, n\}, \end{cases} \tag{68}$$

which, combined with (67), implies that, for all $k \in \{n - m, \ldots, n\}$ and $\delta u \in (-\varepsilon_0, \varepsilon_0)$,

$$F(k, \overline{u}_{\rm eq} + \delta u) = \sum_{i=n-k}^{m} (f_i(\overline{u}_{\rm eq}) + b_i\delta u) + \sum_{i=n-k}^{m} r_i(\delta u),$$

which implies that, for all $k \in \{n - m, \ldots, n\}$ and $\delta u \in (-\varepsilon_0, \varepsilon_0)$,

$$F(k, \overline{u}_{\rm eq} + \delta u) = \sum_{i=n-k}^{m} (f_i(\overline{u}_{\rm eq}) + b_i\delta u) + O(\delta u^2). \tag{69}$$

Since, in addition, $f_m(\overline{u}_{\rm eq}) = 0$, it follows that

$$F(n-m, \overline{u}_{\rm eq} + \delta u) = b_m\delta u + O(\delta u^2). \tag{70}$$

Since b_m is nonzero, it follows from (18), (68), and (70) that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that, for all nonzero $\delta u \in (-\varepsilon_1, \varepsilon_1)$, $d(u) = n - m$, and thus it follows from (70) with $k = n - m$ that, for all $\delta u \in (-\varepsilon_1, \varepsilon_1)$,

$$\mathrm{sign}(F(n-m, \overline{u}_{\rm eq} + \delta u)) = \mathrm{sign}(b_m\delta u). \tag{71}$$

Next, using (6) and (67) implies that, for all $\delta u \in (-\varepsilon_1, \varepsilon_1)$,

$$\sum_{i=0}^{m} f_i(\overline{u}_{\rm eq} + \delta u) = \overline{b}\delta u + O(\delta u^2).$$

Since $\sum_{i=0}^{m} b_i \neq 0$, it follows that there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for all $\delta u \in (-\varepsilon_2, \varepsilon_2)$,

$$\mathrm{sign}\left(\sum_{i=0}^{m} f_i(\overline{u}_{\rm eq} + \delta u)\right) = \mathrm{sign}\left(\overline{b}\delta u\right). \tag{72}$$

Using (71) and (72), it follows from (28) that, for all nonzero $\delta u \in (-\varepsilon_2, \varepsilon_2)$,

$$\begin{aligned}
&\operatorname{sign}(h(\overline{u}_{\mathrm{eq}} + \delta u)) \\
&\quad= \operatorname{sign}\left(F(n-m, \overline{u}_{\mathrm{eq}} + \delta u)\sum_{i=0}^{m} f_i(\overline{u}_{\mathrm{eq}} + \delta u)\right) \\
&\quad= \operatorname{sign}\left(b_m \delta u\right) \operatorname{sign}\left(\overline{b}\delta u\right) \\
&\quad= \operatorname{sign}\left(b_m \overline{b}\delta u^2\right). \qquad (73)
\end{aligned}$$

Next, define the monic polynomial $N(z) \triangleq \sum_{i=0}^{m} \frac{b_i}{b_m} z^i$. Then, it follows from (73) and *iv*) that, for all nonzero $\delta u \in (-\varepsilon_2, \varepsilon_2)$, $b_m^2 \delta u^2 N(1) < 0$, which implies that $N(1) < 0$. It thus follows from [14, Lemma S1] that $N(z)$ has at least one root outside of the closed unit disk.

It follows from *v*) that the explicit ZD (5) is satisfied in a neighborhood of $\overline{u}_{\mathrm{eq}}$. Using (55) and the chain rule to linearize (5) about $\overline{u}_{\mathrm{eq}}$ yields, for all $k \geq m-n$, the linearized ZD

$$\begin{aligned}
\delta u_{k+1} &= \left.\frac{\mathrm{d} f_m^{-1}(u)}{\mathrm{d}u}\right|_{u=\sum_{i=1}^{m-1} -f_i(\overline{u}_{\mathrm{eq}})} \\
&\quad \cdot \sum_{i=1}^{m-1} -f_i'(\overline{u}_{\mathrm{eq}})\delta u_{k-m+i+1} \\
&= \frac{\sum_{i=1}^{m-1} -f_i'(\overline{u}_{\mathrm{eq}})\delta u_{k-m+i+1}}{f_m'(\sum_{i=0}^{m-1} -f_i(\overline{u}_{\mathrm{eq}}))} \\
&= \frac{1}{b_m}\sum_{i=0}^{m-1} -b_i \delta u_{k-m+i+1}. \qquad (74)
\end{aligned}$$

Multiplying (74) by b_m implies that, for all $k \geq m-n$,

$$\sum_{i=0}^{m} b_i \delta u_{k-m+i+1} = 0. \qquad (75)$$

With k replaced by $k-(n-m+1)$, (75) implies that, for all $k \geq 1$, the linearized ZD is given by

$$\sum_{i=0}^{m} b_i \delta u_{k-n+i} = 0, \qquad (76)$$

whose characteristic polynomial is $b_m N(z)$. Since $N(z)$ has at least one root outside of the closed unit disk, it follows that $\delta u \equiv 0$ is an unstable equilibrium of (76). Therefore, Lyapunov's indirect method implies that $\overline{u}_{\mathrm{eq}}$ is an unstable ZDE of the DIH system (1). □

If, for all $i = 0, \ldots, m$, the input function f_i is linear, then Theorem 3 is related to the linear results given by [14]. In particular, for all $i = 0, \ldots, m$, define $f_i(u) \triangleq \beta_i u$, where $\beta_m \neq 0$ and $\sum_{i=0}^{m} \beta_i \neq 0$. In this case, *ii*) and *iii*) of Theorem 3 are satisfied. If *iv*) of Theorem 3 is also satisfied, then the argument used in the proof of Corollary 2 implies that $N(1) < 0$, where $N(z) \triangleq \sum_{i=0}^{m} \frac{b_i}{b_m} z^i$. It thus follows from [14, Lemma S1] that $N(z)$ has an odd number of real roots greater than 1. Since $N(z)$ is the characteristic polynomial of the linear ZD, it follows that 0 is an unstable ZDE of the DIH system (1) with linear input functions.

Example 10: This example illustrates Theorem 3 by showing that i)–v) imply that $\overline{u}_{\mathrm{eq}} \equiv 0$ is an unstable ZDE. For the DIH system (43), it follows from (14) that $F(1, u) = -u$, which, together with (18), implies that, for all nonzero $\overline{u}$, $d(\overline{u}) = 1$. Therefore, (28) implies that, for all nonzero $\overline{u}$,

$$h(\overline{u}) = F(1, \overline{u})\sum_{i=0}^{1} f_i(\overline{u}) = -\overline{u}^2\left(\overline{u}^2 + 0.1\right) < 0. \qquad (77)$$

In this case, Theorem 1 implies that, for all nonzero $\overline{u}$, the step response of (43) has initial undershoot.

Next, note that the unique ZDE is $\overline{u}_{\mathrm{eq}} = 0$. Note that, since $f_1(u) = -u$ is bijective, it follows from (5) that, for all $k \geq -1$, the explicit ZD of (43) are given by

$$u_{k+1} = \left(u_k^2 + 1.1\right) u_k. \qquad (78)$$

Hence, for all $k \geq 1$, $|u_k| \geq 1.1^k |u_0|$, which, assuming that $u_0 \neq 0$, implies that $|u_k| \to \infty$ as $k \to \infty$. ⋄

V. CONCLUSION

For discrete-time, input-output Hammerstein (DIH) systems, which have linear unforced dynamics and nonlinear zero dynamics (ZD), this paper showed that instability of the ZD is a consequence of initial undershoot for all sufficiently small step heights; this result provides a discrete-time converse of results given in [6] and [15]. In view of the critical role of zeros in systems and control, it is clear that a deeper and more comprehensive understanding of the properties of unstable ZD is warranted. In particular, a continuous-time LTI system has initial undershoot if and only if the system possesses an odd number of positive NMP zeros, while initial undershoot occurs in a discrete-time LTI system if and only if the system possesses an odd number of positive zeros greater than unity. *The analogous conditions for nonlinear systems are unknown. This open problem is of theoretical interest as well as practical importance.* Along the same lines, the relationship between initial undershoot and ZD within the context of normal forms [2], [13] is an open problem.

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